10/21/2026 Fall 2026, Proofs and Problem Solving I Test 2 Solutions Sitaraman
Howard University Math Department

1. (20 points) Prove that the set of rational numbers $\mathbb{Q}$ is countable by writing it as a countable union of countable sets. In other words, write $\mathbb{Q} = S_1 \cup S_2 \cup .... \cup S_n \cup ....$ where each S_i is countable. They need not be disjoint. You may use the fact that countable union of countable sets is countable, without proof.

   We can write for a fixed denominator $i \in \mathbb{N}$, $S_i = \{a/i \mid a \in \mathbb{Z}\}$ then each S_i is countable, with a one-to-one map to the integers $a/i \rightarrow a$. Since the integers are countable, S_i is countable also.

   Then we get a countable union of countable sets as the denominators vary over all natural numbers and they cover all the rational numbers.

   Thus the set of rational numbers is countable.

2. (20 points) State whether countable or uncountable. You must prove your answer using the facts proved so far. (You can use the facts that the real numbers are uncountable and rational numbers are countable, without proof).

   1. The set of points $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the xy-plane) such that $a^2 + b^2 < 1$.

   2. The set of real numbers with only a finite number of digits. (terminating decimal expansion).

   Solution:

   Proof:

   1. The set of points $(a, b) \in \mathbb{R} \times \mathbb{R}$ the xy-plane such that $a^2 + b^2 < 1$ is simply all the points inside the disk of radius 1 and center at (0,0). This is uncountable because it contains all the real numbers corresponding to the points $(x, 0)$ in the interval $(-1, 1)$. If $-1 < x < 1$ then $x^2 + 0^2 = x^2 < 1$. We saw in class $(-1, 1)$ can be shown to be in bijection with all real numbers using the tangent function. So if the set of points inside the disk were countable, the points in $(-1, 1)$ which is a subset, would also be countable. This contradiction proves that the points in the disk have t obe uncountable.

   2. Countable: These are all rational numbers. For example: 1.34567 is just 134567/100000.

3. (15 points) Prove the following: If $X = S_1 \cup S_2 \cup .... \cup S_n$ is a partition of a set X into disjoint union of the subsets $S_1, S_2, ..., S_n$ then R defined by " xRy iff x, y belong to the same subset S_i for some i" is an equivalence relation on X. Prove also that the equivalence classes of R are the subsets $S_1, S_2, ..., S_n$.

   Solution:

   Proof: This was proved in class and it is in the notes / the textbook.

4. For the relation R defined on the set of integers $\mathbb{Z}$ by $mRn \iff m+n = 2k$ for some integer k, prove the following:

   a) (10 points) Prove that it is an equivalence relation.

   (b) (6 points) Find the equivalence class of 1. Describe the set using set notation.

   (c) (8 points) How many equivalence classes are there? Describe all of them using set notation.

   (d) (6 points) Show that the equivalence classes are distinct and they cover all integers.

   Solution: (You can also prove this by saying $m+n=2k$ means m, n are both odd or both even).

   Reflexive: $\forall m \in \mathbb{Z}, m+m = 2m$ so mRm is true.

   Symmetric: $\forall m, n \in \mathbb{Z}, mRn \implies nRm$ because $m+n=2k \implies n+m=2k$.

   Transitive: $\forall m.n, k \in \mathbb{Z}, mRn, nRk \implies mRk$ because $m+n=2k_1, n+k=2k_2 \implies m+2n+k = 2(k_1+k_2) \implies m+k = 2(k_1+k_2-n)$ and $k_1+k_2-n \in \mathbb{Z}$.

   (b) $[1] = \{m \in \mathbb{Z} \mid mR1\} = \{m \in \mathbb{Z} \mid m+1 = 2k\}$. This means all m such that $m = 2k-1 \implies m = 7k+1$. So the set is all odd integers $\{...., -5, -3, -1, 1, 3, 5, ...\}$.

   (c) and (d) Similar to [1], you will have [0] equals all even integers $\{...., -4, -2, 0, 2, 4, ...\}$.. The two classes are disjoint and together cover all integers.

5. (15 points) Check if the following relation R defined on the set of real numbers S is reflexive, symmetric, and transitive:

   "a is related to b iff $a-b$ is irrational."

   Solution:

   Reflexive: $aRa \implies a-a = 0 \notin \mathbb{Q}$. This is not true because 0 is rational.

   Symmetric: $aRb \implies bRa$ is true because $x-y \notin \mathbb{Q} \implies y-x = -(x-y) \notin \mathbb{Q}$. Otherwise both will be rational.

   Transitive: aRb and bRc do not necessarily imply aRc.

   Counterexample: $1-\sqrt{2}$ and $\sqrt{2}-3$ are irrational but $1-3$ is rational.

6. (Challenge: Extra credit 10 points) What is wrong with the following proof?

   Every real number has a countable number of decimal places.

   Each decimal place has a finite (actually 10) and thus countable number of digits.

   A countable union of countable sets is countable.

   So the set of real numbers is countable.

   Solution:

The problem is that this confuses the cartesian product with union. Cartesian product is like a set of vectors, with each vector having the same number of coordinates. You can think of the real numbers as a Cartesian product but it is not a union of the numbers in each digit position.

If you have a countable cartesian product of countable sets, say $A_1 \times A_2 \times A_3 \times .... \times A_n \times .....$ and each element looks like $(a_1, a_2, a_3, ..., a_n, ...)$ then just as when we showed the real numbers are uncountable, we can choose $b_1 = 1$ if $a_1 \neq 1$ and $b_1 = 2$ if $a_1 = 1$ (or something similar) and then $(b_1, b_2, b_3, ..., b_n, ...)$ cannot be in any ordered listing of elements of the Cartesian product.