Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 50 minutes

Please read the questions carefully before answering

1. (20 points) Prove that the set of rational numbers $\mathbb{Q}$ is countable by writing it as a countable union of countable sets. In other words, write $\mathbb{Q} = S_1 \cup S_2 \cup .... \cup S_n \cup ....$ where each S_i is countable. They need not be disjoint. You may use the following facts without proof:
   1. Countable union of countable sets is countable, without proof.
   2. The set of integers is countable.

2. (20 points) State whether countable or uncountable. You must prove your answer using the facts proved so far. (You can use the facts that the real numbers are uncountable and rational numbers are countable, without proof).
   1. The set of points $(a, b) \in \mathbb{R} \times \mathbb{R}$ (the xy-plane) such that $a^2 + b^2 < 1$.
   2. The set of real numbers with only a finite number of digits. (terminating decimal expansion).

3. (15 points) Prove the following: If $X = S_1 \cup S_2 \cup .... \cup S_n$ is a partition of a set X into disjoint union of the subsets $S_1, S_2, ..., S_n$ then R defined by " xRy iff x, y belong to the same subset S_i for some i" is an equivalence relation on X. Prove also that the equivalence classes of R are the subsets $S_1, S_2, ..., S_n$.

4. For the relation R defined on the set of integers $\mathbb{Z}$ by $mRn \iff m+n = 2k$ for some integer k, prove the following:

   a) (10 points) Prove that it is an equivalence relation.

   (b) (6 points) Find the equivalence class of 1. Describe the set using set notation.

   (c) (8 points) How many equivalence classes are there? Describe all of them using set notation.

   (d) (6 points) Show that the equivalence classes are distinct and they cover all integers.

5. (15 points) Check if the following relation R defined on the set of real numbers S is reflexive, symmetric, and transitive:

   "a is related to b iff $a - b$ is irrational."

6. (Challenge: Extra credit 10 points) What is wrong with the following proof?

   Every real number has a countable number of decimal places.

   Each decimal place has a finite (actually 10) and thus countable number of digits.

   A countable union of countable sets is countable.

   So the set of real numbers is countable.