

1. (20 points) State the converse, contrapositive and negative of the following statement about functions taking real numbers to real numbers :

If the function $f(x)$ is continuous in $[a, b]$ then it is integrable in $[a, b]$.

Converse: If the function $f(x)$ is integrable in $[a, b]$ then it is continuous in $[a, b]$.

Contrapositive: If the function $f(x)$ is not integrable in $[a, b]$ then it is not continuous in $[a, b]$.

Negative: The function $f(x)$ is continuous in $[a, b]$ but it is not integrable in $[a, b]$.

2. (20 points) Use truth table to check that $p \wedge \neg q$ and $(\neg p) \vee q$ are negatives of each other.

Solution:

p	q	$p \wedge \neg q$	$\neg p \vee q$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	F	T

The truth values being opposite in each case, shows that the two statements are negatives (opposites) of each other.

3. (20 points) Prove the following statements or give counterexample if false:
 - a) If x, y are irrational real numbers, then $x + y$ is irrational.
 - b) Every continuous real valued function on real numbers function has a maximum (peak) or minimum (bottom) on the real number line.

Solution:

a) False. $\sqrt{2}, -\sqrt{2}$ are irrational but their sum equals 0 and is rational.

b) False. The function $y = 2x$ doesn't have a peak or a bottom.

4. (20 points) Prove using the contrapositive or contradiction: You must first state the contrapositive.
 1. If $x + y + z > 30$ then one of x, y, z is greater than 10.
 2. If n does not have a divisor $d_1 \geq \sqrt{n}$ (other than n itself) then it does not have a divisor $d \leq \sqrt{n}$ (other than 1).

Solution:

Proof:

1. Contrapositive: If all of x, y, z are less than or equal to 10, then their sum is also less than or equal to 10. You can see it by adding.

NOTE: "One of them is greater than 10" doesn't mean "Only one of them" or "Exactly one of them." If that were the case, it would say so. As mentioned in class, when we say in math that x or y or z is something, it doesn't mean only one of them. In general in math, you assume only what is explicitly stated, not more than that.

Anyway, this time if you did assume that it said "only one of them" I haven't deducted too many points.

2. Contrapositive: If an integer n has a divisor $d \leq \sqrt{n}$ (other than 1) then it has a divisor $d_1 \geq \sqrt{n}$ (other than n itself) .

Proof: If $d \leq \sqrt{n}$ then $n/d \geq n/\sqrt{n} = \sqrt{n}$. So n/d is the required divisor.

5. (20 points) Prove that the composition of two surjective functions is surjective :

$$f : A \rightarrow B \text{ is onto}, g : B \rightarrow C \text{ is onto} \implies g \circ f : A \rightarrow C \text{ is onto}.$$

Solution:

We want to show that $z \in C \implies \exists x \in A \ni g \circ f(x) = z$.

Since g is onto, $\exists y \in B \ni g(y) = z$.

Since f is onto, $\exists x \in A \ni f(x) = y$.

Then $g \circ f(x) = g(f(x)) = g(y) = z$.

6. (Extra credit 10 points) Prove using contradiction that cube root of 4, denoted $\sqrt[3]{4}$, is irrational.

You may need the following fact about integers: a prime number p divides ab , i.e $p|ab$ means $p|a$ or $p|b$.

Solution:

Assume $\sqrt[3]{4}$ is rational, let $\sqrt[3]{4} = m/n$, with m, n being integers with no common factors.

Then $4 = m^3/n^3$ and we get $4n^3 = m^3$. This means 2 divides m^3 . But 2 being a prime, it divides a product only if it divides one of the numbers. So it has to divide m because m^3 is a product of m by itself three times. Let $m = 2k$. Plugging this back into $4n^3 = m^3$ we get $4n^3 = 8k^3$. Cancelling 4 we get $n^3 = 2k^3$. Using same argument as before we get $2|n$. Thus 2 is a common factor of m and n , which contradicts our assumption. So $\sqrt[3]{4}$ is irrational.