10-27-2026 Notes, Proofs and Problem Solving

Proof by Induction

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Outline

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Proof by Induction: Where do we use it?

Induction is useful when you need to prove something in an infinite number of cases.
For example, to prove that $n^2 + n$ is always even, you need to show that $1^2 + 1, 2^2 + 2, 3^2 + 3, \ldots$ and so on forever are all even!

Even proving a million cases is not enough.

1. First prove statement for $n = 1$.
2. Next assume it true for $n = k$.
3. Using the previous case, prove for $n = k + 1$.

KEY POINT: SHOWING THAT FROM ANY STEP WE CAN GO TO NEXT STEP

Proof of countability of subsets

The proof of countability of subset S of natural numbers (or any countable set) uses well-ordering theorem and induction.
BASE CASE : Using the theorem, get least natural number s_1 in S.
INDUCTION HYPOTHESIS: Assume we have ordered n elements $s_1, s_2, ..., s_n$ of S in ascending order.
INDUCTIVE STEP: Use the well-ordering theorem again to get least element of $S - \{s_1, s_2, ..., s_n\}$.
This is s_{n+1} and so we have arranged the first $n+1$ elements in ascending order.
The inductive step shows we can keep arranging the elements of S like this, forever.

Proof of infinitude of primes

The proof of existence of infinite number of primes can be done using induction.
BASE CASE : The first prime p_1 is 2.
INDUCTION HYPOTHESIS: Assume we have the first n primes $p_1, p_2, ..., p_n$ in ascending order.
INDUCTIVE STEP: Need to find next prime p_{n+1}.
Assume it doesn't exist. Then $P = (p_1 p_2 .... p_n) + 1$ cannot be divisible by any of $p_1, p_2, ..., p_n$ because they leave a reminder of 1 upon dividing P. Moreover P is clearly bigger than all of those primes. So P has to be a prime number itself, which is a contradiction. So p_{n+1} exists.
The inductive step shows we can keep finding new primes like this, forever.

Proof of infinitude of primes – second proof

Alternate proof, also by induction.
BASE CASE : The first prime p_1 is 2.
INDUCTION HYPOTHESIS: Assume we have the first n primes $p_1, p_2, ..., p_n$ in ascending order.
INDUCTIVE STEP: Need to find next prime p_{n+1}.
Let $P = (p_1 p_2 ....p_n) + 1$. P cannot be divisible by any of $p_1, p_2, ..., p_n$ because they leave a reminder of 1 upon dividing P. Either it is prime or it is divisible by some prime q that is different from $p_1, p_2, ..., p_n$. If P is prime let $p_{n+1} = P$ otherwise let q be p_{n+1}. The inductive step shows we can keep finding new primes.
NOTE: When we said “Either it is prime or it is divisible by some prime q" we are assuming that all integers can be factored uniquely. This needs to be proven. Proof later.

Example problem A : solution by induction

Problem A: Prove directly : If $n = 4m$ is a natural number (that is a multiple of 4), then 2^{3m} is a factor of $n!$.
The smallest example: $2^{3(1)} = 2^3$ is a factor of $(4 \times 1)! = 4! = 1 \times 2 \times 3 \times 4 = 24$.
ONE WAY TO EXPRESS THIS:
DIVIDE $n! = (4m)!$ INTO m BLOCKS OF 4 NUMBERS:

$$(4m)! = (1 \times 2 \times 3 \times 4) \times ..... \times (4m-3 \times 4m-2 \times 4m-1 \times 4m)$$

$$= (2 \times 4 \times (1 \times 3)) \times ... \times (4m-3 \times 2(2m-1) \times 4m-1 \times 4(m))$$

$$= (8 \times 2) \times ... \times (8 \times (4m-3 \times 2m-1 \times 4m-1 \times m))$$

$$= \underbrace{(8 \times 8 \times ... \times 8)}_{m \text{ times}} \times (2 \times 210 \times .... \times (4m-3 \times 2m-1 \times 4m-1 \times m))$$

$= 8^m \times (an\ integer)$. So $8^m = (2^3)^m = 2^{3m}$ is a factor of $n!$.

Example problem A : solution by induction – contd

The solution above can be proved using induction:

1. First prove statement for $m = 1$, where m is chosen suitably.
   This was done in the question itself!
2. Next assume it true for $m = k$.
   In other words, 2^{3k} divides $(4k)!$.
   Then $(4k)! = 2^{3k}M$ for some natural number M.
3. Using the previous case, prove for $m = k + 1$.
   This means we need to show that
   $4(k+1)! = (4k+4)! = 2^{3(k+1)}(\textit{some number})$ or that
   $2^{3(k+1)}$ divides $4(k+1)! = (4k+4)!$.

Example problem A : solution by induction – the last step

For this, look at
$4(k+1)! = (4k+4)! = (4k+4)(4k+3)(4k+2)(4k+1)(4k)!$.
You can write this as $4(k+1)(4k+3)2(2k+1)(4k+1)(4k)!$.
Separating the powers of 2 and plugging in $(4k)! = 2^{3k}M$,
$4(k+1)! = (4k+4)! = 8(k+1)(4k+3)(2k+1)(4k+1)(2^{3k})M = 2^3(2^{3k})(\textit{some number})$.
But $2^3 2^{3k} = 2^{3k+3} = 2^{3(k+1)}$.
So finally we get $4(k+1)! = (4k+4)! = 2^{3(k+1)}(\textit{some number})$ which means that $2^{3(k+1)}$ divides $4(k+1)! = (4k+4)!$.

NOTICE HOW WE ARE USING $(4k)! = 2^{3k}M$ FROM $m = k$ TO PROVE FOR $m = k+1$.

Exercise 1: Proof that $n^2 + n$ is even by induction

Prove that $n^2 + n$ is even by using induction.
Proof:
Step 1: Prove for $n = 1$. $1^2 + 1 = 2$ is even, so true for $n = 1$.
Step 2: Assume true for $n = k$. So we will use that $k^2 + k$ is even to prove the case when $n = k + 1$.
Step 3: Prove for $n = k + 1$. We need to show $(k + 1)^2 + (k + 1)$ is even.
But $(k + 1)^2 + (k + 1) = k^2 + 2k + 1 + k + 1 = (k^2 + k) + (2k + 2) = (k^2 + k) + 2(k + 1)$.
Now $k^2 + k$ is even by assumption, and $2(k + 1)$ is an even number, AND the sum of two even numbers is also even, so $(k + 1)^2 + (k + 1)$ is also even.

NOTICE HOW WE ARE USING THAT $k^2 + k$ IS EVEN, FROM THE $m = k$ STEP TO PROVE FOR $m = k + 1$.

Exercises 2 and 3

In the questions below n represents a natural number.

2. Prove that for all natural numbers $3^n - 1$ is always even by using induction.

3. Prove that $n! \geq 2^{n-1}$ is always true by using induction.

Exercise 2 solution

2. Prove that for all natural numbers $3^n - 1$ is always even by using induction.

This could actually be done quickly without induction. All you need to do is to note that product of odd numbers is always odd, so any power of 3 is odd, and so that minus 1 is even.

Proof using induction:
First show for $n = 1$. We get $3^1 - 1 = 2$ is even, which is true.

Next we need to show that we can go from any k to $k + 1$.
In other words, if $3^k - 1$ is even for any of $1, 2, 3, \ldots$ up to infinity, then $3^{k+1} - 1$ is also even.
This is the key step and is done in next slide.

Exercise 2 solution: From k to $k+1$

For this first we note that $3^{k+1}-1 = 3(3^k)-1$.
So if we want to use the even-ness of 3^k-1 then we need to connect it to 3^k and then we can plug it into
$3^{k+1}-1 = 3(3^k)-1$.
But $3^k = 3^k-1+1$!! So plugging this for 3^k we get

$$3^{k+1}-1 = 3(3^k)-1 = 3(3^k-1+1)-1 = 3(3^k-1)+3-1 = 3(3^k-1)+2.$$

So we have written the statement from the $k+1$-th step in a way that lets us use the statement from the k-th step.
So $3^{k+1}-1 = 3(3^k-1)+2 = 3(\textit{even number})+2$. But 3 times an even number is also even. Then even number plus 2 is even. So we have proved that, IF 3^k-1 is even, THEN $3^{k+1}-1$ is also even.

Exercise 3 solution

3. Prove that $n! \geq 2^{n-1}$ is always true by using induction.

Again, start with $n = 1$: We have $1! = 1 \geq 2^{1-1} = 2^0 = 1$.
Next we need to show that we can go from any k to $k + 1$.
In other words, if $k! \geq 2^{k-1}$ for any of $1, 2, 3, \ldots$ up to infinity, then $(k + 1)! \geq 2^{(k+1)-1} = 2^k$ also.
This is the key step and is done in next slide.

Exercise 3 solution: From k to $k+1$

To go from $k!$ to $(k+1)!$ first we note that $(k+1)! = (k+1)(k!)$.
If we want to use that $k! \geq 2^{k-1}$ to prove
$(k+1)! \geq 2^{(k+1)-1} = 2^k$ then we need to go from $k!$ to $(k+1)!$.
Now in order to do this without changing the inequality
$k! \geq 2^{k-1}$ we multiply BOTH sides by $k+1$.

$$k! \geq 2^{k-1} \implies (k+1)k! \geq (k+1)2^{k-1} \implies (k+1)! \geq (k+1)2^{k-1}.$$

Now it is true that $k+1 \geq 2$ for all $k = 1, 2, 3, \ldots$ So

$$(k+1)! \geq (k+1)2^{k-1} \implies (k+1)! \geq 2(2^{k-1}) = 2^k$$

Thus we have proved that, IF $k! \geq 2^{k-1}$ THEN $(k+1)! \geq 2^k$.

Exercises 4, 5, 6 and 7

In the questions below n represents a natural number. It will be useful to try a few cases to get an idea or identify a pattern.

4. Recall how we proved that a set of n elements has 2^n subsets (see notes from 9/18 and 9/23, pages 10-12). Prove the same using induction.

5. Prove that a polygon with n sides will have $n(n-3)/2$ diagonals.

6. Prove that the n-th derivative of x^m with $n \leq m$ is

$$\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}(x^{m-n}).$$

7. Prove that the n-th derivative of e^{kx} is $\frac{d^n}{dx^n}(e^{kx}) = k^n e^{kx}$.

Answer for Exercise 4

4. Prove that a set of n elements has 2^n subsets using induction.

Solution: If the set has only one element, i.e, $n = 1$, then there are two subsets: the empty set and itself. So $|P(X)| = 2^1 = 2$. (Actually we need to start with empty set. In this case there is only one subset, and $|P(X)| = 2^0 = 1$).
Suppose it is true for n.
So assuming any set with n elements has 2^n subsets we need to show that any set with $n + 1$ elements has 2^{n+1} subsets.

Answer for Exercise 4 – contd.

If X has $n+1$ elements, fix the $n+1-$th element, say x and divide $P(X)$ into two kinds of subsets:
All the subsets not containing x are basically the subsets of a set of n elements, namely the first n. So there are 2^n such subsets based on the assumption for n.
All subsets containing x can be obtained by adding x to one of the subsets in the previous collection, and each one in the previous collection gives rise to exactly one subset in this new collection. In other words, they are in one to one correspondence and hence there are 2^n subsets in the new collection also.
But every subset either contains x or doesn't. So total number of subsets is obtained by adding number of subsets in each collection. So $P(X) = 2^n + 2^n = 2(2^n) = 2^{n+1}$.

Another way to think of Proof by Induction.

Another way to think of proof by induction is :
We are trying to prove a pattern that exists in all cases $n = 1, 2, 3, ...$
In the case of the previous problem, the pattern is that every time we add an element, we double the number of subsets.
In addition to subsets of the the first n elements, we add new subsets by putting the $n + 1$-th element in each of the 2^n subsets of the set with n elements.
For example, when you go from $\{ 1 \}$ to $\{1, 2 \}$:
First we had two subsets $\{ \}, \{ 1 \}$
By putting 2 in each of these, we get two new subsets $\{ 2 \}, \{ 1, 2 \}$.
Thus there are totally four subsets for $\{1, 2 \}$.

Answer for Exercise 5

The smallest polygon is the triangle with 3 sides, and it has no diagonals.
The formula gives $n(n-3)/2 = 3(3-3)/2 = 0$. So it works for $n = 3$.
Now what is the pattern as we go from n to $n+1$?
By looking at a few examples (always a good idea), we see that each time we add a new side to the polygon with n sides we add $n-1$ diagonals.
This observation seems to agree with what happens when we go from n to $n+1$ in the formula:

$$\frac{(n+1)((n+1)-3)}{2} - \frac{n(n-3)}{2} = \frac{n^2-n-2}{2} - \frac{n^2-3n}{2} = n-1.$$

Why is this the case? Proving this pattern is really what needs to be done now.

Answer for Exercise 5 – contd.

The proof of the pattern as we go from n to $n+1$:
When we add one more side to the polygon with n sides, you can see that we are also adding a new vertex.
Now this vertex gives you $n-2$ new diagonals by connecting to every other vertex in the n-gon except the ones next to it on the left and on the right.
In addition to these $n-2$ diagonals we get an extra vertex because one side (the one between the two vertices next to the new vertex) becomes a diagonal now.
Therefore we get $(n-2)+1 = n-1$ new diagonals.

Definition of sequences and exercises 8,9,10,11

A sequence is a set of numbers (could be real or even complex) ordered by the natural numbers.
You may say that it is the image of a function on the natural numbers.
For example, the sequence of even numbers $2, 4, 6, \ldots$ is given by $a_n = 2n$ for $n = 1, 2, 3, \ldots$.

Write a formula for the $n-$th term of the following and prove it by induction:

Exercise 8. $3, 7, 11, 15, \ldots\ldots$

Exercise 9. $1, 5, 25, 125, \ldots$

Exercise 10. $3, \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \ldots\ldots\ldots$

Exercise 11. $-2, 4, 16, 256, \ldots$

Answer for Exercise 8

We see that each time we are adding 4.
A sequence where same number is added each time is called an **arithmetic sequence**.
So for the n-th term we would have added $n-1$ of the 4's.
So $a_n = 3 + (n-1) \times 4 = 3 + 4n - 4 = 4n - 1$.

Proof that $a_n = 4n - 1$:
For $n = 1$ it works because $4(1) - 1 = 3$.
Suppose it works for n. how do we get to $n+1$?
In other words, how do we prove $a_{n+1} = 4(n+1) - 1 = 4n + 3$?
We know that we get a_{n+1} by adding another 4.
So we must have $a_{n+1} = a_n + 4 = (4n - 1) + 4 = 4n + 3$.
So we have showed that, if $a_n = 4n - 1$, then
$a_{n+1} = 4n + 3 = 4(n+1) - 1$.

Exercise 12: From Fall 2018 Midterm Exam

Exercise 12: Find a formula for the n-th term a_n of the sequence

$$1, \frac{2!}{2^2}, \frac{3!}{3^3}, \ldots.$$

Using induction show that $a_n \leq 1$ for all natural numbers n.
Solution: (a) $a_n = n!/n^n$
(b) $a_n \leq 1 \implies n!/n^n \leq 1 \implies n! \leq n^n$.
This is proved by induction in example 2 of Section 1.2.3 in Raji's Introduction to Number Theory ebook.
Proof continued below.

Exercise 12: From Fall 2018 Midterm Exam - solution

One way to prove that $a_{n+1} \leq 1$ using $a_n \leq 1$:

$$a_n = \frac{n!}{n^n} \leq 1 \implies \frac{n!(n+1)}{n^n(n+1)} \leq 1$$

$$\implies \frac{n!(n+1)}{(n+1)^n(n+1)} \leq 1 \implies \frac{(n+1)!}{(n+1)^{n+1}} \leq 1.$$

In the above derivation

$$\frac{n!(n+1)}{n^n(n+1)} \leq 1 \implies \frac{n!(n+1)}{(n+1)^n(n+1)} \leq 1$$

is true because

$$n^n < (n+1)^n \implies \frac{1}{(n+1)^n} \leq \frac{1}{n^n}.$$

Solution to exercises 6 and 7

6. Prove that the n-th derivative of x^m with $n \leq m$ is $\frac{d^n}{dx^n}(x^m) = \frac{m!}{(m-n)!}(x^{m-n})$.

Solution: First note that m is fixed, and also that if $n > m$ then the n-th derivative is 0. For example, 2nd and higher derivatives of the linear function $x^1 = x$ are all 0.

For $n = 1$ this is just the power rule:

$$\frac{d^1}{dx^1}(x^m) = \frac{d}{dx}(x^m) = \frac{m!}{(m-1)!}(x^{m-1}) = mx^{m-1}.$$

If the formula for n is true, we need to show that what we get by plugging in $n+1$ is, in fact, the $n+1$-th derivative.

$$i.e, \frac{d^{n+1}}{dx^{n+1}}(x^m) = \frac{m!}{(m-(n+1))!}(x^{m-(n+1)}) = \frac{m!}{(m-n-1)!}(x^{m-n-1}).$$

Solution to exercise 6 (cont.d)

To see this we must get the $n+1$-th derivative FROM the n-th derivative. But this, by definition, is just the derivative of the n-th derivative:

$$\frac{d^{n+1}}{dx^{n+1}}(x^m) = \frac{d}{dx}\left(\frac{d^n}{dx^n}(x^m)\right) = \frac{d}{dx}\left(\frac{m!}{(m-n)!}(x^{m-n})\right)$$

$$= \frac{m!}{(m-n)!}(m-n)(x^{m-n-1}) = \frac{m!}{(m-n-1)!}(x^{m-n-1}) \text{ because}$$

$(m-n)! = (m-n)(m-n-1)(m-n-2)....3.2.1 =$
$(m-n)((m-n-1)!)$.

Solution to exercise 7

7. Prove that the n-th derivative of e^{kx} is $\frac{d^n}{dx^n}(e^{kx}) = k^n e^{kx}$.

Proof: For $n = 1$ we have $\frac{d^1}{dx^1}(e^{kx}) = \frac{d}{dx}(e^{kx}) = ke^{kx} = k^1 e^{kx}$ using chain rule.

If the formula for n is true, we need to show that what we get by plugging in $n + 1$ is, in fact, the $n + 1$-th derivative.

i.e, $\frac{d^{n+1}}{dx^{n+1}}(e^{kx}) = k^{n+1}(e^{kx})$.

Solution to exercise 7 –cont.d

To see this we must get the $n+1$-th derivative FROM the n-th derivative. But this, by definition, is just the derivative of the n-th derivative:

$$\frac{d^{n+1}}{dx^{n+1}}(e^{kx}) = \frac{d}{dx}\left(\frac{d^n}{dx^n}(e^{kx})\right)$$

$$= \frac{d}{dx}\left(k^n(e^{kx})\right) = k^n\frac{d}{dx}(e^{kx}) = k^n(ke^{kx}) = k^{n+1}e^{kx}.$$

Arithmetic and Geometric Sequences

ARITHMETIC SEQUENCE:
Go from a_n to a_{n+1} by ADDING a fixed number d, called **the common difference**

GEOMETRIC SEQUENCE:
Go from a_n to a_{n+1} by MULTIPLYING a fixed number r, called **the common ratio**

Example of Arithmetic Sequence: What is d?

$$-11, -3, 5, 13, ...$$

Example of Geometric Sequence: What is r ?

$$3, 4, \frac{16}{3}, \frac{64}{9}, ...$$

Sum of first n terms of the Arithmetic Sequence 1,2,3,...

The sum of first n natural numbers is $f(n) = n(n+1)/2$.

Proof by Induction:
Clearly it is true for $n = 1$.
Suppose it is true for n. i.e, the sum of first n natural numbers is $f(n) = n(n+1)/2$.
Then we need to show that
$f(n+1) = (n+1)((n+1)+1)/2 = (n+1)(n+2)/2$ is the sum of the first $n+1$ natural numbers.

Contd: Sum of first n terms of the Arithmetic Sequence 1,2,3,...

But the sum of the first $n+1$ natural numbers is

$$(1+2+3+....+(n-1)+n)+n+1 = f(n)+n+1 = n\left(\frac{n+1}{2}\right)+(n+1)$$

$$= (n+1)(\frac{n}{2}+1) = (n+1)(n+2)/2 = f(n+1).$$

Exercises in summing sequences

13. Find a pattern (and a formula) for the sum of the first n odd numbers. Can you explain the formula geometrically?

14. Find a pattern (and a formula) for the sum of the first n numbers of $3, 7, 11, 15, \ldots$.

Exercises in summing sequences – Solution for 13 and 14

The pattern for ANY arithmetic sequence is that the sum can be found by adding the same sequence twice, writing it in reverse order on the second sum.

$$\begin{array}{l} 1 \quad +3 \quad +5 \quad +\ldots..+(2n-3)+(2n-1) \\ \underline{(2n-1)+(2n-3)+(2n-5)+\ldots..+3+\quad +1} \\ =2n \quad +2n \quad +2n\ldots\ldots\ldots\ldots\ldots+2n \quad +2n \end{array}$$

$= 2n \times n = 2n^2$.

Since this is 2 times the original sum, original sum is n^2.

Notice that the sum of the first term (namely 1) and the last term (namely $2n-1$) is repeated n times. Similarly for any arithmetic sequence, the sum will be found by multiplying n times the sum of first and last terms and then dividing by 2.

Sum of first n terms of any Arithmetic Sequence

Prove by induction that the sum of the first n terms of an arithmetic sequence starting with a and adding d each time is $n(2a+(n-1)d)/2$.

Proof: Let $S(n) = n(2a+(n-1)d)/2$.
Clearly it is true for $n = 1$: $S(1) = a = 1(2a+0d)/2 = a$.
Suppose it is true for n. i.e, the sum of first n terms is $S(n) = n(2a+(n-1)d)/2$.
Then we need to show that
$S(n+1) = (n+1)(2a+(n+1-1)d)/2 = (n+1)(2a+nd)/2$
is the sum of the first $n+1$ terms.

Contd: Sum of first n terms of any Arithmetic Sequence

But the sum of the first $n+1$ terms is

$$a + (a+d) + ... + (a+(n-1)d) + (a+(n+1-1)d)$$

$$= [a + (a+d) + ... + (a+(n-1)d)] + (a+(n+1-1)d)$$

$$= S(n) + (a+nd) = n\left(\frac{2a+(n-1)d}{2}\right) + (a+nd)$$

$$= n\left(a + \left(\frac{(n-1)d}{2}\right)\right) + (a+nd) = a(n+1) + n(\frac{n-1}{2} + 1)(d)$$

$$= a(n+1) + n\left(\frac{n+1}{2}\right)d = (n+1)(2a+nd)/2.$$

Sum of first n terms of any Geometric Sequence

Prove by induction that sum of a geometric series is

$$\sum_{k=1}^{k=n} ax^{k-1} = a\left(\frac{x^n - 1}{x - 1}\right)$$

Proof: Clearly true for $n = 1$.
Suppose true for n. i.e, suppose we have

$$\sum_{k=1}^{k=n} ax^{k-1} = a\left(\frac{x^n - 1}{x - 1}\right)$$

We need to show this for $n + 1$. i.e,

$$\sum_{k=1}^{k=n+1} ax^{k-1} = a\left(\frac{x^{n+1} - 1}{x - 1}\right)$$

Contd: Sum of first n terms of any Geometric Sequence

As in problem 3 this is the sum of the first n terms plus the $n+1$-th term.

$$a\left(\frac{x^n-1}{x-1}\right)+ax^n = a\left(\frac{(x^n-1)+(x-1)x^n}{x-1}\right)$$

$$= a\left(\frac{x^{n+1}-1}{x-1}\right)$$

Answer for exercise 11

First we find the n-th term of the sequence $-2, 4, 16, 256, \ldots$.
You notice a pattern: each term is a square of the previous one.
[In fact, this is an example of recursive sequence: $a_{n+1} = f(a_n)$, as are the arithmetic and geometric sequences].
Each time you square, the power increases by a power of 2.
In other words, if $n > 1$ and $a_n = (-2)^{2^{n-1}}$ then
$a_{n+1} = (-2)^{2^{n-1+1}} = (-2)^{2^n}$.
how do we express and prove this pattern using induction?
We just use the pattern we observed in the beginning!

$$a_{n+1} = (a_n)^2 = \left((-2)^{2^{n-1}}\right)^2 = (-2)^{2^{n-1} \times 2} = (-2)^{2^n}$$

Exercises 15,16,17,18

Prove the following using induction:

15. $$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)}, \ n = 1, 2, ...$$

16. (Longer) $2^n \geq n^2, \ n = 4, 5, ...$

17. $7^n - 1$ is divisible by 6 for all $n \geq 1$.

18. (Harder)

$$\frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)} \leq \frac{1}{\sqrt{n+1}}, \ n = 1, 2, ...$$

Answer for Exercise 15

Prove the following using induction:

$$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)}, \ n = 1, 2, ...$$

For $n = 1$ we get $1/2 = 1/2$. For $n + 1$ we get

$$\frac{1}{2(n+1)} \leq \frac{1 \times 3 \times 5 \times .... \times (2n-1) \times (2(n+1)-1)}{2 \times 4 \times 6 \times ... \times (2n) \times 2(n+1)}$$

We need to get this from the inequality for n. To do this, notice that the RHS of the n-th step is contained in the RHS of the $n+1$-th step:

$$\frac{1}{2(n+1)} \leq \left(\frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)} \right) \times \frac{(2n+1)}{(2n+2)}$$

Answer for Exercise 15 –cont.d

From the n-th step we have

$$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)}$$

Upon multiplying by $(2n+1)/(2n+2)$ on both sides we get *part* of what we need for $n+1$-th step:

$$\frac{1}{2n}\left(\frac{2n+1}{2n+2}\right) \leq \frac{1 \times 3 \times 5 \times .... \times (2n-1)}{2 \times 4 \times 6 \times ... \times (2n)} \times \left(\frac{2n+1}{2n+2}\right)$$

This will be give us the $n+1$-th step if we show that $\frac{1}{2n+2} \leq \frac{1}{2n}\left(\frac{2n+1}{2n+2}\right)$ but this is true because $\frac{1}{2n}\left(\frac{2n+1}{2n+2}\right) = \frac{1}{2n+2}\left(\frac{2n+1}{2n}\right) \geq \frac{1}{2n+2}$ because $(2n+1)/(2n) \geq 1$ for all $n = 1, 2, 3, ...$

Answer for Exercise 16

16. Prove by induction: $2^n \geq n^2,\ n = 4, 5, ...$

Again for the first step we have $n = 4$ and $2^4 = 4^2$.
We need to use $2^n \geq n^2$ to get $2^{n+1} \geq (n+1)^2$.
Writing $2^{n=1} = 2^n \times 2 = 2^n + 2^n$ and $(n+1)^2 = n^2 + 2n + 1$ we get $2^n + 2^n \stackrel{?}{\geq} n^2 + 2n + 1$.
From n-th step we have $2^n \geq n^2$.

So this reduces to $2^n \stackrel{?}{\geq} 2n + 1$ for all n?
This is also true but needs to be proven, by induction or otherwise.
In fact we have proved something like this before.
The answer involves using the fact that, on RHS we add 2 each time whereas on LHS we multiply by 2 each time.

Some slightly challenging problems

19 . Show that $3^n + 7^n - 2$ is always divisible by 8 for $n \geq 1$.

20 . Show that if you draw n lines none of whom are parallel and no three of whom intersect at a point, then it divides the plane into $(n^2 + n + 2)/2$ disjoint regions.