PROOFS NOTES 10/9/26

Equivalence relations and equivalence classes

FACT: Any equivalence relation R on a set S partitions the set into disjoint equivalence classes. Conversely, any partition of the set S into disjoint subsets gives rise to an equivalence relation R defined by aRb iff a,b belong to same subset A of the partition.

Proof of converse (for the proof of original statement see notes):

Reflexive: aRa for all a in S ? But a belongs to same subset as a. So yes.

Symmetry: If a, b belong to same subset, then b, a belong to same subset.

Transitive: a, b belong to subset A, b and c belong to subset A, then a, c belong to A also.

Note that b cannot belong to any subset other than A (because the subsets are disjoint).

The equivalence classes under this relation are just the subsets.

Proof: [x] = { y in X | xRy } = { y in X | x, y belong to S_n for some n}

= { y in X | y in same S_n as x } = S_n

Example: Let **R** be the set of real numbers and R be the relation on **R** x **R** given by (a,b)R(c,d) iff a+d = b+c.

This can also be written as a-b = c-d. Basically two points are related if the coordinates differ by the same amount.

Reflexive: (a,b) R (a,b) for all points (a,b) in **R** x **R ?** Yes, a-b = a-b.

Symmetry: (a,b) R (c,d) then (c,d) R (a,b) ? a-b = c-d => c-d = a-b ? Yes.

Transitive: Given (a,b), (c,d) and (e,f). a-b = c-d AND c-d = e-f => a-b = e-f => (a,b)R(e,f) if (a,b)R(c,d) and (c,d)R(e,f).

So R is an equivalence relation. What are the equivalence classes?

Example: Equivalence class of (7, 0) = ? (x,y)R(7,0) => x-y = 7-0 = 7. So the points are all in the line y = x-7. Slope is 1 and y-intercept is -7.

In general, equivalence class of (a,b) = Set of points on the line y = x-(a-b) = x-a+b.

SO the set of equivalence classes is just a bunch of lines of slope 1 and y-intercept b-a.

So they cover the whole plane into disjoint subsets.

Question (Classwork 10/19/26): R is a relation on P(S) (subsets of the set S)

A and B are related if A is itself a subset of B. A could equal B.

Is this an equivalence relation?

Reflexive : Any subset is related itself? Yes. A is contained in A because it is equal to itself.

Symmetric: Not true, unless S is empty. A could be contained in B but not vice versa.

Transitive: True. A contained in B and B contained in C means A is contained in C.

Example for equivalence class: R is an equivalence relation defined on the pairs of natural numbers by

(m, n) R (k, l) iff m/n = k/l.

What is the equivalence class of 1 or [(1,1)] = { (m,n) | (m,n)R (1,1) } = {(m,m) | m is a natural number}.

Equivalence class of (1,2) = { (m,n) | (m,n) R (1,2) } = { (m,n) | n = 2m }.