PROOFS NOTES 10/14/26

Union of two countable sets S and T is countable

Proof: The bijection from **N** to S be given by f(n) and the bijection from **N** to T be g(n).

Then define a bijection from **N** to $S \cup T$ by sending odd numbers n to $f(\frac{n+1}{2})$ and even numbers to $g\left(\frac{n}{2}\right)$. This will cover all of S and all of T in a disjoint way because each map has every natural number as input. So it is a surjective map from **N to** $S \cup T$ and by theorem 8.10 that is enough to get a bijection.

Classwork 1: Show that if 3 sets R, S, T are countable, then $R \cup S \cup T$ is countable.

Classwork 2: Give another proof that the set of integers is countable by writing it as a union of 3 sets : Positive and negative integers and 0.

Uncountability of real numbers

Enough to show that the real numbers in (0,1) are uncountable.

Suppose they are countable, and can be written as $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots.$

Let $a_1 = 0.a_{11}a_{12}a_{13} \ldots\ldots\ldots\ldots\ldots\ldots$

$a_2 = 0.a_{21}a_{22}a_{23} \ldots\ldots\ldots\ldots$

.....

$a_n = 0.a_{n1}a_{n2}a_{n3} \ldots\ldots\ldots\ldots\ldots$

.....

Let $b = 0.b_1b_2b_3 \ldots\ldots b_n \ldots.$

Where $b_i = 2\ if\ a_{ii} = 1\ and\ b_i = 1\ if\ a_{ii} \neq 1$

This is the "diagonal" argument. Can you see that b cannot be in original list $a_1, a_2, a_3, \ldots\ldots, a_n, \ldots.$?

Uncountability of Irrational Numbers

Proof: Suppose the set of irrationals I is countable.

Then the real numbers **R** = **I U Q** is also countable. That contradicts the uncountability of real numbers.

Challenge: Show that the power set P(S) doesn't have same cardinality as S.

P(S) is the set of all subsets of S.

NOTE: If S is finite, then P(S) is also finite, with 2^n elements where n = |S|

Idea of proof: Suppose there is a surjective map S -> P(S).

Define a subset T = { x in S such that x is not in f(S) }

Show that T cannot be in f(S).