PROOFS NOTES 10/16/26

PROBLEMS ON COUNTABILITY AND CARDINALITY

Problem 8.10: If S is denumerable, then S is equinumerous with a proper subset of itself.

Proof: Denumerable means there is a bijection of S with **N**.

In other words, you can order the elements of the set as s_1, s_2, s_3, ...

NOTE: The set has to be infinite.

f : **N** -> S and f is a bijection. Let T be a proper subset of S.

Then T is countable because S is countable.

So there is a bijection g: T -> **N**.

Now we get a bijection between T and S by taking the composition of f and g which takes x in T to f(g(x) in S.

Classwork 1: If you have an ordering of a set S, then it is countable.

Let S = { s_1, s_2, s_3, ..., s_n, ...} be the ordering of S. Basically it gives a first element, second element and so on.

Can use the theorem : If there is a surjection **N** -> S then S is countable.

Here the surjection is given by n -> s_n.

FACT: Countable union of countable sets is countable.

Proof in textbook.

Idea similar to trick we used to order the rational numbers.

S_11 s_12 s_13 s_14 .....

S_21 s_22 s_23 s_24 .....

S_31 s_32 s_33 s_34 .....

..... The ordering is obtained by going in a zig zag path, where each diagonal segment of the path has s_ij with the same i+j.

So once we get an ordering we have a countable set, as proved above.

Classwork 2 (Problem 8.9a) Show that the set of polynomials with integer coefficients is countable.

First we can say that cartesian product of **Z** with itself n times is countable.

Given by **Z x Z x ...x Z** (n times) is countable. For instance use the map

(m_1, m_2, ..., m_n) -> 2^(m_1) x 3^(m_2) x .....x p^(m_n) where p is the n-th prime and then use the uniqueness of prime factorization to get an injective map to **N** and then by theorem 8.10, it is bijective with **N**.

So we can associate polynomials with n coefficients (degree n-1) with elements in **Z x Z x ...x Z** (n times).

So let S_n be the set of polynomials with n coefficients (degree n-1).

For each degree n from 0 to infinity we get a countable set S_n.

Just showed that each S_n is a countable set.

Now using the theorem (countable union of countable sets is countable) we get that S_1 U S_2 U S_3 U ... U S_n U..... is also countable.

Classwork 3: (Problem 8.9 b, c) Show that set of algebraic numbers (solutions of polynomials with integer coefficients) is countable. Are the transcendental numbers countable?

Algebraic numbers examples: Sqrt(3) is irrational but it is an algebraic number. It is a solution of the equation x^2-3 = 0

Similarly 1+sqrt(2) is a solution of a quadratic polynomial.

(Just FYI, it is (x-1-sqrt(2))(x-1+sqrt(2)) = x^2-2x+1-2 = x^2-2x-1=0).

For each polynomial with degree n, we get at most n solutions

(fundamental theorem of algebra).

Let P_1 be the solutions of the first polynomial in the ordering of the set of polynomials of integer coefficients. Similarly P_2, P_3, and so on.

We get a countable union of countable (finite) sets. So it is countable.

(Actually even if there are infinitely many solutions for a polynomial, this will work, as long as they are countable).

Transcendental numbers are real numbers that are not algebraic.

Example is Pi. Pi is not a solution of any polynomial with integer coefficients.

Transcendental numbers are not countable because if they were, then the real numbers would be countable because they will be the union of two countable sets : the algebraic and the transcendental.