11/19/2025 Fall 2025, Proofs and Problem Solving I Test 3 v2 Solutions Sitaraman

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1. (20 points) Prove using induction that $2^{n+1} < 1 + (n+1)2^n$ for all $n \in \mathbb{N}$.

   For $n = 1$ we have $2^2 < 1 + (2 \times 2) = 5$.

   Assume true for n.

   Then $2^{n+1} < 1 + (n+1)2^n$.

   Multiply both sides by 2. Get $2^{n+2} < 2 + (n+1)2^{n+1}$.

   But the RHS is $< 1 + (n+2)2^{n+1}$ because $2 + (n+1)2^{n+1} = 1 + (n+1)2^{n+1} + 1$.

   Proof that $1 + (n+1)2^{n+1} + 1 < 1 + (n+2)2^{n+1}$: (work backwards):

   $1 + (n+1)2^{n+1} + 1 < 1 + (n+2)2^{n+1} \implies 1 < 2^{n+1}$ and this is true for all positive integers.

   So $2^{n+2} < 1 + (n+2)2^{n+1}$ which is the statement for $n+1$.

2. (Base 3 expansion) Prove using strong induction that every positive integer n is either a power of 3 or can be written as the sum of powers of 3 with coefficients $1, 2$, or 0.. In other words, $n = b_0 + b_1 3 + b_2 3^2 + ... + b_k 3^k$ for some positive integer k and the b_i are 1,2, or 0.

   Solution:

   Try a few examples: $1 = 3^0, 2 = 2, 3 = 3^1, 4 = 1 + 361, 5 = 2 + 3^1, ...$

   We see that it seems to be true for all the small cases, and we can increase by 1 or by 2 or by $3 = 1 + 2$ and so on.

   So the base case can be just 1 and it is a power of 3.

   Assume statement is true for all k with $1 \leq k < n$. Need to prove it for n.

   If n is itself a power of 3 we are done.

   If not, let divide n by 3 and get $n = 3m + b_0$ where $b_0 = 1, 2$ or 0. Actually, b_0 is just the first digit in the base 3 expansion of n.

   Clearly m is smaller than n and so $m = b_1 + b_2 3 + b_3 3^2 + ... + b_k 3^j$ by the induction hypothesis.

   So $n = b_0 + 3(b_1 + b_2 3 + b_3 3^2 + ... + b_j 3^j) = b_0 + 3b_1 + 3^2 b_2 + ... + b_j 3^{j+1}$. So n is a sum of powers of 3, as required.

3. (20 points) Prove using basic definition of limit that if s_n, t_n are convergent and $\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty}(s_n - t_n) = s - t$.

   Solution: Proof is in textbook.

4. (20 points) Prove that the recurrence sequence given by $s_1 = 1, s_n = \sqrt{4s_n + 1}$ is monotonic increasing, and bounded. Without finding the limit, explain why the limit should exist.

Solution:

Increasing: (Proof by induction) $s_2 = \sqrt{4+1} = 5$ and $s_2 > s_1 = 1$.

Assume $s_{n-1} > 0$ and $s_{n-1} \leq s_n$. Multiplying by 4 and adding 1 doesn't change direction of inequality because they are positive. Then take square root of both sides, also without changing inequality for same reason.

$$4s_{n-1} + 1 \leq 4s_n + 1 \implies \sqrt{4s_{n-1} + 1} \leq \sqrt{4s_n + 1} \implies s_n \leq s_{n+1}.$$

Bounded: We have proved $s_n \leq s_{n+1}$ for ALL n. So $s_n \leq \sqrt{4s_n + 1}$ for all n. Squaring (again, direction of inequality remains same) we get

$$s_n^2 \leq 4s_n + 1 \implies s_n^2 - 4s_n - 1 \leq 0 \implies (4 - \sqrt{20})/2 \leq s_n \leq (4 + \sqrt{20})/2.$$

So sequence is always bounded by $2 + \sqrt{5}$.

The limit should exist because it is a bounded, increasing sequence of real numbers. The monotone convergence theorem guarantees existence of least upper bound and we showed in class that it is the limit.

5. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough. Each 5 points.

a) The limit of the quotient of any two sequences will equal the quotient of the limits.

(b) For a function to be continuous at $x = c$ it is enough if the function is defined at $x \to c$.

(c) A bounded, increasing sequence of irrational numbers always has a irrational number as least upper bound.

(d) For any sequence of real numbers $x_n \to c$, with c being any real number, $\tan x_n \to \tan c$.

Solution:

a) False. This is true only if all the sequences are convergent. Example: $s_n = n, t_n = n^2$. Their quotient goes to 0 but you cannot define product of the limits because they don't exist.

(b) False. The limit should exist and also equal the value at c.

(c) False. It has a real number least upper bound but not necessarily irrational. For example, take any rational number like 2 and look at the sequence $2 - (\pi/n)$.They are all irrational (prove!) and smaller than 2 and their limit (and least upper bound) is 2 as $n \to \infty$.

(d) False. If the real number is $\pi/2$ then tan function is undefined.

6. (Challenge, extra credit 20 points) Prove using the definition of limits that every real number is a limit of a sequence of rational numbers of the form $a_i/10^{k_i}$ where a_i is an integer and k_i is a non-negative integer. In other words, decimal expansion exists for all real numbers. (You cannot start with a decimal expansion! Goal is to prove it exists).

   Solution: Let r be any real number and let $N \leq r$ be the greatest integer $\leq r$. Now define $r - N = r_1$. Then $0 \leq r_1 < 1$. (Why can't $r - N$ equal 1?)

   Now find a positive integer k_1 such that $10^{k_1} r_1 \geq 1$. Let N_1 be the greatest integer $\leq 10^{k_1} r_1$. Let $r_2 = 10^{k_1} r_1 - N_1$. Then as before $0 \leq r_2 < 1$.

   Now we have $r = N + r_1 = N + (N_1 + r_2)/10^{k_1}$.

   Let $a_1 = 10^{k_1} N + N_1$. Then $r = \frac{a_1}{10^{k_1}} + \frac{r_2}{10^{k_1}}$. We have

$$\left| r - \frac{a_1}{10^{k_1}} \right| = \frac{r_2}{10^{k_1}} < \frac{1}{10^{k_1}} \text{ because } r_2 < 1.$$

   Continuing this process by induction, we can get a_i and k_i such that

$$\left| r - \frac{a_i}{10^{k_i}} \right| < \frac{1}{10^{k_i}} < \epsilon \text{ for any } \epsilon > 0 \text{ for some } k_i.$$