**Howard University Math Department**

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 50 minutes

Please read the questions carefully before answering

1. (20 points) Let $a_n = \dfrac{n!}{n^n}$. Prove using induction that $a_n \leq 1$ for all $n \in \mathbb{N}$.

2. (Binary expansion) Prove using strong induction that every positive integer $n$ is either a power of 2 or can be written as the sum of distinct powers of 2. In other words, $n = 2^{a_1} + 2^{a_2} + ... + 2^{a_k}$ and $a_1, a_2, ..., a_k$ are all different, and $a_i$ are non-negative integers.

3. (20 points) Prove using basic definition of limit that if $s_n, t_n$ are convergent and $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$, then $\lim_{n\to\infty}(s_n + t_n) = s + t$.

4. (20 points) Prove that the recurrence sequence given by $s_1 = 1, s_n = \sqrt{3s_n + 1}$ is monotonic increasing, and bounded. Without finding the limit, explain why the limit should exist.

5. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough. Each 5 points.

   a) The limit of the product of any two sequences will equal the product of the limits.

   (b) For a function to be continuous at $x = c$ it is enough if the limit of the function exists as $x \to c$.

   (c) A bounded, increasing sequence of rational numbers always has a rational number as least upper bound.

   (d) For any sequence of real numbers $x_n \to c$, with $c$ being any real number, $\ln x_n \to \ln c$.

6. (Challenge, extra credit 20 points) Prove using the definition of limits that every real number is the limit of a sequence of rational numbers of the form $a_i/10^{k_i}$ where $a_i$ is an integer and $k_i$ is a non-negative integer. In other words, decimal expansion exists for all real numbers. (You cannot start with a decimal expansion! Goal is to prove it exists).