10/17/2025 Fall 2025, Proofs and Problem Solving I Test 2 v1 Solutions Sitaraman

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1. (15 points) Prove directly that A is a subset of B iff $A \cap B$ equals A.

   Start by stating the condition for any two sets to be equal. Then use that condition in your proof of above statement.

   To show that $A \cap B$ equals A we need to show each is a subset of the other.

   $A \subseteq B \implies A \cap B = A$:

$$\text{“}A \subseteq B\text{”} \implies \text{“}x \in A \implies x \in B\text{”} \implies \text{“}A \subseteq A \cap B\text{”}$$

$$\text{“}x \in A \cap B \implies x \in A\text{”} \implies A \cap B \subseteq A.$$

$$\text{“}A \cap B \subseteq A\text{” } \textit{and} \text{ “}A \subseteq A \cap B\text{”} \implies \text{“}A = A \cap B\text{”}$$

   Converse: If $A \cap B$ equals A then $A \subseteq B$.

   Suppose not. Then there is $x \in A$ such that $x \notin B$. This also means there is $x \in A$ such that $x \notin A \cap B$. So $A \cap B$ does not equal A.

2. (a) (10 points) Show that $aRb \iff a - b \in \mathbb{Z}$ (the integers) is an equivalence relation on the real numbers $\mathbb{R}$. Note that $a, b \in \mathbb{R}$.

   (b) (5 points) Find equivalence class of 0.

   Solution:

   Reflexive: $a - a = 0 \in \mathbb{Z}$ for any real number a. So aRa for all $a \in \mathbb{R}$.

   Symmetric: $a - b \in \mathbb{Z} \iff b - a = -(a - b) \in \mathbb{Z}$. So aRb means bRa for all $a, b \in \mathbb{R}$.

   Transitive: $a - b \in \mathbb{Z}$ *and* $b - c \in \mathbb{Z} \implies a - c \in \mathbb{Z}$ because sum of two integers is also an integer. SO aRb and bRc imply aRc.

   Equivalence class of 0: $a \in (0) \implies aR0 \implies a - 0 \in \mathbb{Z} \implies a \in \mathbb{Z}$.

   So the equivalence class of 0 is the set of integers.

3. (15 points) Let X be the set of real numbers on the real number line.

   Show that $aRb \iff a \leq b$ is NOT an equivalence relation.

   What conditions are satisfied and what conditions are not satisfied?

   Solution: It is reflexive because $a \leq a$ is always true.

   It is NOT symmetric because $a \leq b$ does not mean $b \leq a$ unless a, b are equal.

   It is transitive because if $a \leq b, b \leq c$ then $a \leq c$.

4. (10 points) Prove by cases: if 2 divides $x + y$ then 2 divides $x - y$.

   Solution:

   x and y have to be either both odd or both even, in order for $x + y$ to be even. (In the other two cases you get $x + y$ is an odd number). But in both those cases $x - y$ is even as well.

5. (15 points) Prove using contradiction: $\sqrt[3]{2}$ (Cube root of 2) is irrational.

   Solution:

   Proof: Assume $\sqrt[3]{2}$ is rational. Let $\sqrt[3]{2} = m/n$ with m, n having no common factors. Then cubing both sides we get $2 = m^3/n^3$.

   From this we get $2n^3 = m^3$ which means $2|m^3 \implies 2|m$ due to the prime factorization theorem. Let $m = 2k$ then $m^3 = 8k^3$ and plugging this back into $2n^3 = m^3$ we get $2n^3 = 8k^3 \implies n^3 = 4k^3$. From this we get $2|n^3$ and as before this gives $2|n$.

   NOTE: You can't say 4 divides n.

   But then 2 is a common factor of m, n and we get the contradiction.

6. (15 points) Show that two finite sets have a bijection between them iff they have the same number of elements. (You must prove the statement as well as its converse).

   Solution: Let the two sets be A and B.

   If they have a bijection f between them, then $f(A)$ has the same number of elements as A because f is 1-1. But $f(A) \subseteq B$ and f is onto means $f(A) = B$. This means B has the same number of elements as A, as well.

   On the other hand, if A and B have the same number of elements, we can name the elements $A = \{a_1, a_2, ..., a_n\}$ and $B = \{b_1, b_2, ..., b_n\}$ and say $f(a_i) = b_i$ and this will be a 1-1 onto map.

7. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough.

   a) (5 points) The set of all real numbers with only finitely many digits (positions) in their decimal expansion is a countable set.

   (b) (5 points) The equivalence classes of a set under any equivalence relation are disjoint.

   (c) (5 points) The set of functions from a finite set to itself is countable.

   Solution:

   a) True. This is just a subset of the rational numbers: Any such number can be written as a fraction. Note that it is not all of the rational numbers. 1/3 for example is not in this set, as it has a repeating and hence infinite decimal expansion. We proved in class that subsets of countable set is countable.

(b) True. It was proved in class and proof is in the notes.

(c) True. If a finite set has n elements, then any function can only take those n elements to one of those n elements. So totally there $n \times n \times .... \times n$ (n times) possibilities for functions. So there are only n^n such functions and the set is finite, hence countable.

8. (Challenge problem, extra credit 20 points) This exercise shows that there cannot be a bijection between $\mathbb{N}$ (natural numbers 1,2,3,.....) and its power set $P = \mathcal{P}(\mathbb{N})$, namely the set of all subsets of $\mathbb{N}$. So the power set of $\mathbb{N}$ is uncountable. In general power set of any set has a bigger cardinality (so no bijection between them).

   It is enough to show there no surjection $\mathbb{N} \to P$. Suppose f is a surjection.

   Question : Show that the following subset cannot be in $f(\mathbb{N})$:

$$T = \{x \in \mathbb{N} \ : \ x \notin f(x)\}.$$

   Solution:

   Proof in textbook, chapter 9 Theorem 8.18. (Chapter titled Cardinality).