Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 30 minutes

Please read the questions carefully before answering

1. (15 points) Prove directly that A is a subset of B iff $A \cap B$ equals A.

   Start by stating the condition for any two sets to be equal. Then use that condition in your proof of above statement.

2. (a) (10 points) Show that $aRb \iff a - b \in \mathbb{Z}$ (the integers) is an equivalence relation on the real numbers $\mathbb{R}$. Note that $a, b \in \mathbb{R}$.

   (b) (5 points) Find equivalence class of 0.

3. (15 points) Let X be the set of real numbers on the real number line.

   Show that $aRb \iff a \leq b$ is NOT an equivalence relation.

   What conditions are satisfied and what conditions are not satisfied?

4. (10 points) Prove by cases: if 2 divides $x + y$ then 2 divides $x - y$.

5. (15 points) Prove using contradiction: $\sqrt[3]{2}$ (Cube root of 2) is irrational.

6. (15 points) Show that two finite sets have a bijection between them iff they have the same number of elements. (You must prove the statement as well as its converse).

7. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough.

   a) (5 points) The set of all real numbers with only finitely many digits (positions) in their decimal expansion is a countable set.

   (b) (5 points) The equivalence classes of a set under any equivalence relation are disjoint.

   (c) (5 points) The set of functions from a finite set to itself is countable.

8. (Challenge problem, extra credit 20 points) This exercise shows that there cannot be a bijection between $\mathbb{N}$ (natural numbers 1,2,3,.....) and its power set $P = \mathcal{P}(\mathbb{N})$, namely the set of all subsets of $\mathbb{N}$. So the power set of $\mathbb{N}$ is uncountable. In general power set of any set has a bigger cardinality (so no bijection between them).

   It is enough to show there no surjection $\mathbb{N} \to P$. Suppose f is a surjection.

   Question : Show that the following subset cannot be in $f(\mathbb{N})$:

$$T = \{x \in \mathbb{N} \ : \ x \notin f(x)\}.$$