12/9/2025 Final Exam Solutions Fall 2025,Proofs and Problem Solving Sitaraman

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Instructions: **NO CALCULATORS OR CELLPHONES**

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 120 minutes; Total 100 points.

Please read the questions carefully before answering.

1. (10 points) Prove by cases: For any two real numbers x, y we have

$$max\{x, y\} = \frac{x + y + |x - y|}{2}$$

   Solution: Divide into cases as follows: Either $x \geq y$ or $y \geq x$. If $x \geq y$ then $max\{x, y\} = x$ and $|x - y| = x - y$ because $x - y$ is positive.

   So $\frac{x + y + |x - y|}{2} = \frac{x + y + (x - y)}{2} = x$ and equals $max\{x, y\}$.

   Similarly when $x \leq y$.

2. (20 points) Write the negative, converse and contrapositive of the following statement. Then prove it if it is true, give a counterexample if it is false: "If a function maps the finite set X ONTO the finite set Y, then every element in Y has a unique pre-image in X."

   (every element in Y has unique preimage is same as saying different elements in X have different images).

   Solution: In the following it is understood that X, Y are finite.

   Negative (If A then NOT B) : If a function maps the set X onto the set Y, then some element in Y does not have a unique pre-image in X.

   Converse (If B then A) : If every element in Y has a unique pre-image in X then the function maps the set X onto the set Y.

   Contrapositive (If NOT B then NOT A) : If some element in Y doesn't have a unique pre-image in X then the function does not map the set X onto the set Y.

   The given statement is false. If X has more elements than Y then f can be onto without being one-to-one. For example we can map the set $\{1, 2\}$ onto $\{1\}$ by sending both 1 and 2 to 1 and it won't be 1-1.

3. (10 points) Prove if true or give counterexample:

   For all sets A, B, and C:

   Either $C \subseteq A$ or $C \subseteq B$ if and only if $C \subseteq A \cup B$.

Use the following definition: Given any two sets S, T, $S \subseteq T$ means that each element of S is also in T.

Solution: If $x \in C$ and $C \subseteq A$ or $C \subseteq B$ then $x \in A$ or $x \in B$. Therefore $x \in A \cup B$ and $C \subseteq A \cup B$.

But the converse is not true. It is possible that $C \subseteq A \cup B$ but neither $C \subseteq A$ nor $C \subset B$. For example $\{a, b\} \subset \{a, c\} \cup \{b, c\}$ but $\{a, b\}$ is not a subset of $\{a, c\}$ or $\{b, c\}$.

4. Let R be an equivalence relation on a set A. Let B be the set of equivalence classes of A. Define a function f from A to B by the rule $f(x) = [x]$.

   (a) (6 points) Show that this is an onto function.

   (b) (8 points) Given an example of a set A with an equivalence relation R such that f is a one-one function.

   (c) (6 points) Show that $f(x) = f(y) \iff xRy$.

   Solution:

   a) a) This is well defined because every element x belongs to an equivalence class, namely $[x]$ because R has to be reflexive and xRx has to be true. It is also onto because every equivalence class is equivalence class of some element x in A.

   b) Let $A = \mathbb{R}$, be the set of real numbers. Then define R by $xRy \iff x = y$. Then each element is its own equivalence class, i.e, $[x] = \{x\}$. In this case clearly f is one-one.

   c) $f(x) = f(y) \iff [x] = [y] \iff x \in [y] \text{ or } y \in [x] \iff xRy.$

5. (20 points) Prove by induction for all natural numbers n and a fixed real number x:

   If $2 + x > 0$ then $(2 + x)^n \geq 1 + n + nx$.

   Solution:

   For $n = 1$ we have $2 + x = 1 + 1 + x$ so it works.

   Assume true for n, and prove for $n+1$. In other words, prove $(2+x)^{n+1} \geq 1+(n+1)+(n+1)x$.

   Given $(2 + x)^n \geq 1 + n + nx$ multiply both sides by $2 + x$.

   Get $(2 + x)^n(2 + x) \geq (1 + n + nx)(2 + x)$. Note that we are multiplying both sides by a positive number, so the inequality doesn't change direction.

   Now if we prove that $(1 + nx + n)(2 + x) \geq 1 + (n + 1)x + (n + 1)$ then we will be done. But $(1 + nx + n)(2 + x) = 2 + 2nx + (n + 1)x + nx^2 + 2n$ which can be rewritten as $= 1 + (n + 1)x + (n + 1) + (n + 2nx + nx^2) = 1 + (n + 1)x + n + 1 + n(1 + x)^2$ and this is $\geq 1 + (n + 1)x + (n + 1)$ because $n(1 + x)^2$ is always nonnegative.

   [This is just the Bernoulli's inequality proved in class, namely $1+x > 0 \implies (1+x)^n \geq 1+nx$ with x replaces by $1 + x$].

6. (10 points) Find a bijection (1-1, onto map) from the set of integers $\mathbb{Z}$ to the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$.

Solution: One possible solution:

$$f(n) = \begin{cases} 1 & n = 0 \\ 2n & n > 0 \\ 1 - 2n & n < 0 \end{cases}$$

So $f(0) = 1, f(1) = 2, f(2) = 4, ..., f(-1) = 3, f(-2) = 5, ...$

Easy to see that this is onto. To show it is 1-1, suppose $f(x) = f(y)$. First of all, 0 goes to 1 and none of the others go to 1, because they start at 2 and keep increasing. So we can assume x, y are not 0. Then both x and y must have same sign or else they will go to even and odd or odd and even natural numbers. But if they have the same sign, then either $2x = 2y$ or $1 - 2x = 1 - 2y$ and in both cases $x = y$.

7. (10 points) Using basic definition of limit, show that $\dfrac{n+1}{n} \to 1$ as $n \to \infty$.

   Solution: For any $\epsilon > 0$ e need to find N such that $\left|\dfrac{n+1}{n} - 1\right| < \epsilon$ whenever $n \geq N$.

$$\left|\frac{n+1}{n} - 1\right| < \epsilon \implies |1 + (1/n) - 1| < \epsilon \implies 1/n < \epsilon.$$

   If you choose $N > 1/\epsilon$ this would work.

8. (extra credit 15 points) Prove by contradiction: (You must use basic definitions of limit and continuity, using ϵ, δ etc).

   If a continuous real valued function f has $f(c) > 0$ for $c > 0$ then for some $\delta > 0$ we have $f(x) > 0$ for all x such that $|x - c| < \delta$.

   Solution: Assume for all δ, we have some x in $|x - c| < \delta$ such that $f(x) \leq 0$.

   Then it is not possible that $f(x) \to f(c)$ as $x \to c$.

   Proof: From the assumption we get in every interval $|x - c| < \delta$ some x such that $f(x) \leq 0$. On the other hand we can take $\epsilon > 0$ such that if $|f(x) - f(c)| < \epsilon$ then $f(x) > 0$ also, and because the function is continuous all such x will be in some $|x - c| < \delta$ for some δ. The contradiction means that assumption was wrong.