

Instructions: **NO CALCULATORS OR CELLPHONES**

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 120 minutes; Total 100 points.

Please read the questions carefully before answering.

1. (10 points) Prove by cases: For any natural number  $n$  we have  $n^3 + n$  is even.

Solution: Divide into cases as follows: Either  $n$  is odd or  $n$  is even.

If  $n = 2m + 1$  then  $n^3 + n = (2m + 1)^3 + (2m + 1) = 2(\text{something}) + 2$  which is even.

If  $n = 2m$  then  $n^3 + n = (2m)^3 + (2m)$  which is also even.

2. (20 points) Write the negative, converse and contrapositive of the following statement. Then prove it if it is true, give a counterexample if it is false: "If a function maps the finite set  $X$  1-1 to the finite set  $Y$ , then every element in  $Y$  has a pre-image in  $X$ ."

(every element in  $Y$  has preimage is same as saying function is onto).

Solution: In the following it is understood that  $X, Y$  are finite.

Negative (If A then NOT B) : If a function maps the set  $X$  1-1 to the set  $Y$ , then some element in  $Y$  does not have a pre-image in  $X$ .

Converse (If B then A) : If every element in  $Y$  has a pre-image in  $X$  then the function maps the set  $X$  1-1 to the set  $Y$ .

Contrapositive (If NOT B then NOT A) : If some element in  $Y$  doesn't have a pre-image in  $X$  then the function does not map the set  $X$  1-1 to the set  $Y$ .

The given statement is false. If  $X$  has fewer elements than  $Y$  then  $f$  can be 1-1 without being onto. For example we can map the set  $\{1, 2\}$  onto  $\{1, 2, 3\}$  by sending both 1 and 2 to themselves and it won't be onto because 3 will have no pre-image.

3. (10 points) Prove if true or give counterexample:

For all sets  $A, B$ , and  $C$ :

$C \subseteq A \cup B$  iff either  $C \subseteq A$  or  $C \subseteq B$ .

Use the following definition: Given any two sets  $S, T$ ,  $S \subseteq T$  means that each element of  $S$  is also in  $T$ .

Solution: ( $\Rightarrow$ ) False. It is possible that  $C \subseteq A \cup B$  but neither  $C \subseteq A$  nor  $C \subseteq B$ . For example  $\{a, b\} \subseteq \{a, c\} \cup \{b, c\}$  but  $\{a, b\}$  is not a subset of  $\{a, c\}$  or  $\{b, c\}$ .

But the converse is true. If  $x \in C$  and  $C \subseteq A$  or  $C \subseteq B$  then  $x \in A$  or  $x \in B$ . Therefore  $x \in A \cup B$  and  $C \subseteq A \cup B$ .

4. Let  $R$  be an equivalence relation on a set  $A$ . Let  $B$  be the set of equivalence classes of  $A$ . Define a function  $f$  from  $A$  to  $B$  by the rule  $f(x) = [x]$ .
- (a) (6 points) Show that this is an onto function.
- (b) (8 points) For the equivalence relation on  $\mathbb{N}$  given by  $xRy$  if  $x + y$  is even find the image of 1 and 2. You should describe the elements in each image.
- (c) (6 points) Show that  $f(x) = f(y) \iff xRy$ .

Solution:

a) This is well defined because every element  $x$  belongs to an equivalence class, namely  $[x]$  because  $R$  has to be reflexive and  $xRx$  has to be true. It is also onto because every equivalence class is equivalence class of some element  $x$  in  $A$ .

b) The equivalence classes in  $\mathbb{N}$  are just two subsets the Evens and the Odds. This is because any two even numbers are related because sum of two evens is even and similarly any two odd numbers are related. So the equivalence class of 1 is set of odd natural numbers and that equals  $f(1)$ . Similarly  $f(2)$  is just the set of even natural numbers.

c)  $f(x) = f(y) \iff [x] = [y] \iff x \in [y] \text{ or } y \in [x] \iff xRy$ .

5. (20 points) Prove by induction for all natural numbers  $n$  and a fixed real number  $x$ :

If  $3 + x > 0$  then  $(3 + x)^n \geq 1 + 2n + nx$ .

Solution:

For  $n = 1$  we have  $3 + x = 1 + 2 + x$  so it works.

Assume true for  $n$ , and prove for  $n+1$ . In other words, prove  $(3+x)^{n+1} \geq 1+2(n+1)+(n+1)x$ .

Given  $(3 + x)^n \geq 1 + 2n + nx$  multiply both sides by  $3 + x$ .

Get  $(3 + x)^n(3 + x) \geq (1 + 2n + nx)(3 + x)$ . Note that we are multiplying both sides by a positive number, so the inequality doesn't change direction.

Now if we prove that  $(1 + nx + 2n)(3 + x) \geq 1 + (n + 1)x + 2(n + 1)$  then we will be done. But  $(1 + nx + 2n)(3 + x) = 3 + 3nx + (2n + 1)x + nx^2 + 6n$  which can be rewritten as  $= 1 + (n + 1)x + 2(n + 1) + (4n + 4nx + nx^2) = 1 + (n + 1)x + 2(n + 1) + n(2 + x)^2$  and this is  $\geq 1 + (n + 1)x + 2(n + 1)$  because  $n(2 + x)^2$  is always nonnegative.

[This is just the Bernoulli's inequality proved in class, namely  $1 + x > 0 \implies (1 + x)^n \geq 1 + nx$  with  $x$  replaces by  $2 + x$ ].

6. (10 points) Find a bijection (1-1, onto map) from the the set of natural numbers  $\mathbb{N} = \{1, 2, 3, \dots\}$  to the set of integers  $\mathbb{Z}$ .

Solution: One possible solution:

$$f(n) = \begin{cases} 1 & n = 0 \\ 2n & n > 0 \\ 1 - 2n & n < 0 \end{cases}$$

So  $f(0) = 1, f(1) = 2, f(2) = 4, \dots, f(-1) = 3, f(-2) = 5, \dots$

Easy to see that this is onto. To show it is 1-1, suppose  $f(x) = f(y)$ . First of all, 0 goes to 1 and none of the others go to 1, because they start at 2 and keep increasing. So we can assume  $x, y$  are not 0. Then both  $x$  and  $y$  must have same sign or else they will go to even and odd or odd and even natural numbers. But if they have the same sign, then either  $2x = 2y$  or  $1 - 2x = 1 - 2y$  and in both cases  $x = y$ .

Since this is a bijection it goes both ways:  $\mathbb{Z} \rightarrow \mathbb{N}$  and  $\mathbb{N} \rightarrow \mathbb{Z}$

7. (10 points) Using basic definition of limit, show that  $\frac{n+2}{n} \rightarrow 1$  as  $n \rightarrow \infty$ .

Solution: For any  $\epsilon > 0$  we need to find  $N$  such that  $\left| \frac{n+2}{n} - 1 \right| < \epsilon$  whenever  $n \geq N$ .

$$\left| \frac{n+2}{n} - 1 \right| < \epsilon \implies |1 + (2/n) - 1| < \epsilon \implies 2/n < \epsilon.$$

If you choose  $N > 2/\epsilon$  this would work.

8. (extra credit 15 points) Prove by contradiction: (You must use basic definitions of limit and continuity, using  $\epsilon, \delta$  etc).

If a continuous real valued function  $f$  has  $f(c) > 0$  for  $c > 0$  then for some  $\delta > 0$  we have  $f(x) > 0$  for all  $x$  such that  $|x - c| < \delta$ .

Solution: Assume for all  $\delta$ , we have some  $x$  in  $|x - c| < \delta$  such that  $f(x) \leq 0$ .

Then it is not possible that  $f(x) \rightarrow f(c)$  as  $x \rightarrow c$ .

Proof: From the assumption we get in every interval  $|x - c| < \delta$  some  $x$  such that  $f(x) \leq 0$ . On the other hand we can take  $\epsilon > 0$  such that if  $|f(x) - f(c)| < \epsilon$  then  $f(x) > 0$  also, and because the function is continuous all such  $x$  will be in some  $|x - c| < \delta$  for some  $\delta$ . The contradiction means that assumption was wrong.