9-24-2025 Notes, Proofs 1

Relations, Functions, and Equivalence Relations

Sankar Sitaraman

Math Dept, Howard University

9-24-2025

Outline

What is a relation?

Given a set S and a set T,
a relation can be thought of as a map from S to T.

A lot of times we will talk of a relation *on* a set S.
This will be a map from S to itself.

For example, if S is a set of students, "a knows b" could be thought of a relation from S to S
between students a and b of the same set S.

It is denoted as aRb or $(a, b) \in R$ where R denotes the relation.

Then we could make a map of all the relations, write it as ordered pairs (a, b) or show it in a graph.

Relations as subset of Cartesian Products

aRb is denoted as $(a, b) \in R$ means
the relation R can be thought of as a subset of $S \times T$, the Cartesian Product of S and T.

The Cartesian Product of two sets $S \times T$ is simply the set of all ordered pairs, the first coming from S and the second from T.

For example, the set of points on the plane $\{(x, y) \mid x, y \in \mathbb{R}\}$ is denoted as $\mathbb{R} \times \mathbb{R}$ or $\mathbb{R}^2$.

Example: Relation as subset of Cartesian Products

The map $F : \mathbb{R} \to \mathbb{R}, F(x) = 2x$ is also a relation, represented by the subset $\{(x, 2x) \mid x \in \mathbb{R}\}$.

In other words, a graph is a pictorial representation of a relation, of which a function is a particular case.

In this case, the graph is a line.

Functions : A special kind of relations

A relation F from a set S to a set T is called a **function** if *Every $x \in S$ is assigned to a unique $y \in T$*. Symbolically,

$$\forall x \in S, \exists \; unique \; y \in T \ni F(x) = y.$$

S is called the **Domain** and T is called the **Codomain.**

The set of images $F(x)$ is a subset of T. It is called the **Range.**

Exercise: Find the range of $y = x^2 + 1$ and $y = 2\cos 3x$. Both go from $\mathbb{R}$ to $\mathbb{R}$.

Functions : A special kind of relations – contd

The relation $F : \mathbb{R} \to \mathbb{R}, F(x) = 2x$ is also a function, a map which has a unique image for each element of the domain, in this case $\mathbb{R}$.

F is an example of a **1-1 or injective function.** It maps each x in domain to a different y. Symbolically
$F(x_1) = F(x_2) \implies x_1 = x_2$.
Here $2x_1 = 2x_2 \implies x_1 = x_2$, so it is 1-1.

F is also **onto, or surjective.** Each y in the co-domain has a pre-image. Symbolically $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $y = 2x$. This is true since $x = y/2$ always exists.

$y = x^2 + 1$ from $\mathbb{R}$ to $\mathbb{R}$ is neither 1-1 nor onto.

1-1 and Inverse Functions

If a function $f : A \to B$ is 1-1, it is also **invertible.**

Because each $x \in A$ goes to a different $y \in B$ it is possible to go backwards as well, from $f(A)$ to A. If $y \in f(A)$ meaning $y = f(x)$ for some $x \in A$ then $f^{-1}(y) = x$ and it can only go to one element $x \in A$. (So the inverse map f^{-1} is also a function). Otherwise two elements in A would be mapping to y and f won't be 1-1.

NOTE: We just proved by contrapositive that, if f is 1-1, f^{-1} exists.

For example, $f(x) = 2x$ means $f^{-1}(y) = y/2$.

Composition of Functions

If a function $g : A \to B$ and $f : B \to C$ then $f \circ g(x) = f(g(x))$.

We take the image of g from B and apply f to it.

If f^{-1} exists then $f \circ f^{-1}$ is the identity function.

Exericise: $f(x) = x^2$ and $g(x) = 1/x$ are functions on the real numbers. Write down definitions of $f \circ g(x)$ and $g \circ f(x)$. What is the domain and range of each?
Use: Domain of $f \circ g$ is $x \in \ Dom(g)$ such that $g(x) \in \ Dom(f)$.

Examples of different kind of Functions

The same relation with different domain:
$F : \mathbb{Z} \rightarrow \mathbb{Z}, F(x) = 2x$ is also a function, and it is also **1-1 or injective function.** Proof is the same.

But now F is NOT **onto, or surjective.** Not every y in the co-domain has a pre-image.
In fact only for even numbers, namely $y = 2x$ for some x, pre-image exists since $x = y/2$ is defined in $\mathbb{Z}$.

A relation that is not a Function

A relation that is NOT a function: $\{(y^2, y) \mid y \in \mathbb{R}\}$ is a relation whose graph is a parabola of equation $x = y^2$ with vertex $(0, 0)$ and axis the x-axis.

It is not a function from $\mathbb{R} \to \mathbb{R}$ because not all elements in $\mathbb{R}$ are of the form y^2 or in other words the negative x-axis is not mapped to anything because $y^2 > 0$ always.
Another way to see that it is not a function: For each x, $y = \pm\sqrt{x}$ is not a unique image.

It is a function, though, if domain and codomain are restricted to positive x axis.

A discontinuous map that is a 1-1, but not onto Function

$$f(x) = \begin{cases} x, \ x \leq 0 \\ x+1, \ x > 0 \end{cases}$$

It is not continuous function but it is a function from $\mathbb{R} \to \mathbb{R}$. For each x, either $y = x$ or $y = x + 1$ and in both cases it is a unique image.

It is also 1-1, but it is NOT onto (Prove!)

Testing functions

1. **Vertical Line test:**
   This can be used to test if map given by the graph is a function. If it cuts graph more than once, then for some x we have two or more images, and it is not a function.
2. **Horizontal Line Test**
   This can be used to test if map given by the graph is a 1-1 function. If it cuts graph more than once, then for some y we have two or more pre-images, and it is not a 1-1 function.
   This also tells us if inverse is defined,

Some exercises from textbook

Prove if true, give counterexample if false:

1. If a function $f : A \to B$ and C is a nonempty subset of A then $f(C)$ is a nonempty subset of B.
2. If a function $f : A \to B$ and D is a nonempty subset of B then $f^{-1}(D)$ is a nonempty subset of A.
3. The composition of two surjective functions is always surjective and the composition of two injective functions is always injective.

Some well known and useful functions

1. **Floor and Ceiling Functions:**
   These are defined for all real numbers.
   Floor: $\lfloor x \rfloor$ = Greatest integer smaller than or equal to x.
   Ceiling: $\lceil x \rceil$ = Smallest integer greater than or equal to x.
   Example, $\lfloor 1.5 \rfloor = 1, \lceil 1.5 \rceil = 2$
   The graph of these looks like steps, so these are called **step functions**
2. **Modulus Operator function**
   If $x, y \in \mathbb{Z}$ then $f(x) = x \ mod \ y$ is given by remainder of x when divided by y.
   For example, $10 \ mod \ 4 = 2$.

Equivalence relation on a set

1. Reflexivity: $\forall a \in X,\ aRa$
   (Every element a of X is related to itself)
2. Symmetry: $\forall a, b \in X, aRb \iff bRa$
   (For any two elements a, b of X, a is related to b iff b is related to a).
3. Transitivity: $\forall a, b, c \in X, aRb \text{ and } bRc \implies aRc$.
   (For any three elements $a, b, c \in X$, we have a is related to b and b is related to c means a is related to c).

If all three properties are satisfied by a relation, it is called an "*Equivalence Relation.*"

Some simple examples

1. Is the relation "a knows b" an equivalence relation?
Answer: It is not symmetric or transitive, so it is not an equivalence relation.

2. Given ANY function $f : A \to B$ for any two sets A, B define

$$xRy \iff f(x) = f(y) \quad \forall x, y \in A.$$

This is an equivalence relation on A :
Reflexive: $xRx \; \forall x \in A$ because $f(x) = f(x)$.
Symmetric: $xRy \implies yRx$ because $f(x) = f(y)$ means $f(y) = f(x)$.
Transitive: $xRy, yRz \implies xRz$ because $f(x) = f(y), f(y) = f(z)$ means $f(z) = f(x)$.

Example I

Let $\mathbb{Z}$ be the set of integers. p be a fixed prime number.

Let R be the relation $mRn \iff p|(m-n)$

This means m, n are related if p divides $m-n$.
So $m-n = kp$ for some $k \in \mathbb{Z}$.
Basically m and n leave the same remainder when divided by p.

This is the beginning of "**modular arithmetic**"

This is also called a *congruence* relation.

Check that R is an equivalence relation on $\mathbb{Z}$.

Another way to look at Example I

Let $\mathbb{Z}$ be the set of integers. m be any fixed integer, called the modulus.
The modular relation, as before, is the relation R given by
$xRy \iff m|(x-y)$ or $x-y=km$ for some $k \in \mathbb{Z}$.
As before, it says x, y have same remainder when divided by m.

Let the set $M = \{0, 1, 2, ..., m-1\}$.
When you divide any integer x by m, the remainder can be one and only one of the numbers in M.

Another way to look at R:
Define $f : \mathbb{Z} \to M$ by $f(x) = x_m$, the remainder upon dividing by m. Then R is also defined by: $xRy \iff f(x) = f(y)$.

Example I - the case $p = 2$

Prove that the relation in Example I is an equivalence relation.

Case when $p = 2$.

Reflexive: $\forall m \in \mathbb{Z}, mRm$ because $(m - m) = 0$ which is even. (A number is even is same as saying 2 divides it). Alternatively, every integer leaves same remainder as itself.

Symmetric: $\forall m, n \in \mathbb{Z}, mRn \implies nRm$ because $2|(m - n) \implies 2|(n - m) = -(m - n)$ for any m, n. Alternatively, m leaves same remainder as n is same as saying n leaves same remainder as m.

Transitive: $\forall m, n, k \in \mathbb{Z}, mRn \text{ and } nRk \implies mRk$ because $2|(m - n)$ and $2\ (n - k)$ means $2|(m - n + n - k) = m - k$ for any m, n, k.

Example I - case p is any prime

case p is any prime.

Reflexive: $\forall m \in \mathbb{Z}, mRm$ because $p|(m-m) = 0$ for any m. Alternatively, every integer leaves same remainder as itself.

Symmetric: $\forall m, n \in \mathbb{Z}, mRn \implies nRm$ because $p|(m-n) \implies p|(n-m) = -(m-n)$ for any m, n. Alternatively, m leaves same remainder as n is same as saying n leaves same remainder as m.

Transitive: $\forall m, n, k \in \mathbb{Z}, mRn \text{ and } nRk \implies mRk$ because $p|(m-n)$ and $p\ (n-k)$ means $p|(m-n+n-k) = m-k$ for any m, n, k.
Alternatively, m leaves same remainder as n and n leaves same remainder as k means m leaves same remainder as k.

Equivalence class of an element

Given an equivalence relation R on X,
we can define the equivalence class (a) for any $a \in X$.

$$(a) = \{x \in X \mid aRx\}$$

In other words, equivalence class of a is the set of all elements related to a.

In Example I, with $p = 2$ and $mRn \implies 2|(m-n)$ or $m-n$ is even, the equivalence class of 1 is the set of odd numbers and that of 0 is the even numbers.

In Example I, with $p = 5$ and $mRn \implies p|(m-n)$
what is (1) in $\mathbb{Z}$? What are $(2), (3), (4), (5)$?
Do they cover ALL of $\mathbb{Z}$?

Yes, the "remainder equivalence classes" cover all the integers

In previous slide we asked: In Example I, with $p = 5$ and $mRn \implies p|(m-n)$ what is (1) in $\mathbb{Z}$?

Answer: $(1) = \{...., -14, -9, -4, 1, 6, 11, ...\}$

Notice how all of them leave a remainder of 1 and how they differ from each other by multiples of 5.

Continued: Yes, the "remainder equivalence classes" cover all the integers

What are $(2), (3), (4), (5)$?

$$(2) = \{...., -13, -8, -3, 2, 7, 12, ...\}$$
$$(3) = \{...., -12, -7, -2, 3, 8, 13, ...\}$$
$$(4) = \{...., -11, -6, -1, 4, 9, 14, ...\}$$
$$(5) = \{...., -10, -5, 0, 5, 10, 15, ...\}$$

Notice they do cover ALL of $\mathbb{Z}$.
Note also they are all disjoint! Not a coincidence!
In fact, it is true (see below) that for ANY equivalence relation on ANY set, the equivalence classes will make a disjoint cover of the entire set. **This is called a partition of the set.**

Partitions and Equivalence classes

A BASIC FACT ABOUT EQUIVALENCE RELATIONS ON A SET X

Every equivalence relation on a set X partitions it by the equivalence classes (Proof follows); On the other hand, Any partition of $X = \cup_i X_i$ gives an equivalence relation R for which aRb if $a, b \in X_i$ for some i.
Prove that R is an equivalence relation.

How can you provide an equivalence relation based on being classmates?

Orbits and Equivalence classes

Sometimes **equivalence class of an element is also called its Orbit.**

If f is an invertible function on a set X then you can make an equivalence relation by xRy if $y = f^n(x), n \in \mathbb{Z}$.
Here for positive k we define $f^k = f \circ f \circ .... \circ f$ k times and $f^{-k} = f^{-1} \circ f^{-1} \circ ... \circ f^{-1}$ k times, and f^0 is the identity map sending $x \to x$ for all $x \in X$.

Prove that this is an equivalence relation.
The orbit of $x \in X$ under this map is the set of all images of x.

Orbits and Equivalence classes – example

Sometimes **equivalence class of an element is also called its Orbit.**

What is the orbit of a given integer m under the map $f(x) = x + 1$ on the set of integers $\mathbb{Z}$?

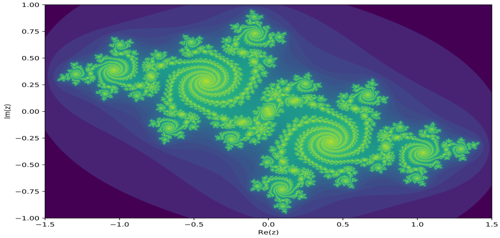
What does it mean to say $y = f^n(x)$?

What is $f^{-1}(x)$?

Orbits and Fractals

A special type of fractals, called **Julia sets** are made using the orbits under the map $f(z) = z^2 + c$ where c is fixed and $z, c \in \mathbb{C}$, the complex numbers.

In the image below, the Julia sets are the central points in yellow (Thanks to Wikipedia)



Example II

The relation R on the set of subsets $X = P(S)$ of $S = \{1, 2, 3, 4, 5\}$ is defined by ARB iff $A \cup Y = B \cup Y$ where $Y = \{4, 5\}$.

Can you show that this is an equivalence relation?

Example II – continued

Relation: ARB iff $A \cup Y = B \cup Y$ where $Y = \{4, 5\}$.

Easy to check that this is an equivalence relation:
Reflexive: $\forall A \in P(S)$, ARA because $A \cup Y = A \cup Y$.

Symmetric: $\forall A, B \in P(S)$, $ARB \implies BRA$ because $A \cup Y = B \cup Y$ is same as $B \cup Y = A \cup Y$.

Transitive: $\forall A, B, C \in P(S)$, ARB & $BRC \implies ARC$ because $A \cup Y = B \cup Y = C \cup Y$.

See next slide for the equivalence classes.

Example II – equivalence classes

Let us find the equivalence class of $A = \{1, 2, 4\}$ just for starters.
If $A \cup Y = B \cup Y$ then $A - Y \subset B \subset A \cup Y$.
In other words, B must contain $A - Y = \{1, 2\}$ and must be itself part of $A \cup Y = \{1, 2, 4, 5\}$.
The possibilities are $\{1, 2\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 2, 4, 5\}$.
Check that each of these are related to A.
Notice how we got one subset for each subset of $Y = \{4, 5\}$ for a total of $2^2 = 4$ subsets.
Now in this equivalence class there is exactly one subset $\{1, 2\}$ that is disjoint from Y, i.e, a subset of $S - Y$.

Example II – equivalence classes – continued

So we saw how in the equivalence class of $\{1, 2, 4\}$ there is exactly one subset $\{1, 2\}$ that is disjoint from Y, i.e, a subset of $S - Y$.
Similarly for each subset of $S - Y$ we will get a different equivalence class.
For instance the equivalence class of $\{1, 3\}$ will be totally disjoint from that of $\{1, 2\}$.
This is because these two subsets cannot be related. If you combine them with Y you get different sets.
But $S - Y = \{1, 2, 3\}$ has $2^3 = 8$ subsets so we will get eight equivalence classes each with exactly four subsets making a total of 32 subsets. But the entire set $P(S)$ has 32 elements in it because S having 5 elements has $2^5 = 32$ subsets.
So in this case also the equivalence classes provide a disjoint

Theorem: Equivalence classes partition the set

Theorem: If R is an equivalence relation on X then X is a *disjoint* union of the equivalence classes.

Proof:
By definition, every element in an equivalence class is an element of X. So $\bigcup_{a \in X} (a) \subseteq X$
On other hand, by reflexivity, every element of X is in its own equivalence class. So $X \subseteq \bigcup_{a \in X} (a)$
So the two sets are equal.
The equivalence classes are disjoint. Proof in next slide:

Theorem: Equivalence classes partition the set – continued

Proof that the equivalence classes are disjoint:
Suppose not.
If an element is in two classes, it means it is related to ALL elements in BOTH of them.
But then by transitivity every element of first set will be related to every element in second set, so the two sets are actually equal!
So either they are completely equal or totally disjoint.

Exercise: example III

Let X be the set of real valued functions on the real number line. Easy to show that $fRg \iff f(0) = g(0)$ is an equivalence relation.

What is the equivalence class of f given by $f(x) = x$?

Note that two functions are related if they cross at the y-axis.

Answer: All functions with $g(0) = 0$ i.e, passing through (0,0), because we need $g(0) = f(0) = 0$.

Exercise: example III continued

A set of representatives can be found by taking all functions of the form $f(x) = x + c$. Each of these has a different $y-$intercept and hence a different point where it crosses y-axis. But at same time, every function crosses the y-axis at some point (or else it does not really have a value at $x = 0$ and thus is not a function of the whole real number line). So every function in X is in the equivalence class of one of these functions $y = x + c$.

Exercise: example IV

Let X be the set of real numbers on the real number line.
Show that $aRb \iff a - b \in \mathbb{Z}$ is an equivalence relation.
For example, 1.2 and 2.2 are related because the difference is -1 which is an integer.

What are the equivalence classes?
Can you provide a set of representatives of distinct equivalence classes that lie in the same interval?

Let S^1 denote the circle of radius one centered at (0,0).
Show that the function $f : \mathbb{R} \to S^1$ given by
$f(r) = (\cos(2\pi r), \sin(2\pi r))$ maps all elements in the same equivalence class to the same point on the unit circle.

More Exercises

Let $\mathbb{R}$ be the set of real numbers on the real number line.
1) Show that $aRb \iff a - b \in \mathbb{Q}$ the rational numbers is an equivalence relation.

What are the equivalence classes?
Can you provide a set of representatives of distinct equivalence classes that lie in the same interval?

2) What if we define $aRb \iff a - b \notin \mathbb{Q}$
3) What about aRb iff a divides b, on the natural numbers $\mathbb{N}$?
4) What about ARB iff $A \subseteq B$ on the set of subsets of a set X?

Countable and Uncountable sets

If there is a bijection $f : \mathbb{N} \to S$ then the set S is called **denumerable.**

A **countable set** is one that is either finite or denumerable

A curious property of infinite sets:

Once can have a bijection from a subset to the whole set!

Show that the set of even numbers E has a bijection with $\mathbb{N}$.

Show that the interval $(0, 1)$ has a bijection with the set of real numbers.

Some Theorems of countable sets

The following theorems are in section titled "Cardinality" (Section 8 in 3rd ed. of Lay's book).
I have modified Theorem 8.10 to make proof more transparent.
Proof of 8.10 uses Theorem 8.9.

8.9 A nonempty subset of a countable set is countable.

8.10 If there is a surjection $\mathbb{N} \to S$ then S is countable

8.10 If there is an injection $S \to \mathbb{N}$ then S is countable

Some properties of countable sets

The following properties can be proved using the theorem 8.10 mentioned above,

1. A subset of a countable set is countable.
2. Union of two countable sets is countable.
3. The Cartesian Product of two countable sets is countable

Some examples of countable and uncountable sets

1. The set of integers is countable.
2. The set of rational numbers is countable.
3. The set of real numbers is uncountable.
4. The set of irrational numbers is uncountable