

PROOFS AND PROBLEM SOLVING

9/10/2025 Class Notes

INTRODUCTION TO SET THEORY

Different kinds of people

- A = Those who know that they know
- B = Those who know that they don't know
- C = Those who don't know that they know
- D = Those who don't know that they don't know

Socrates' Paradox in Set notation

- These sets have no intersection



Notations of set theory

Set builder notation:

Example: $A = \{x \mid x \text{ knows that } x \text{ knows}\}$

Union: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

Intersection: $A \cap B = \{x \in A \text{ and } x \in B\}$

Notations of set theory

Difference: $A - B = \{x \in A | x \notin B\}$

Complement: $\bar{A} = U - A = \{x \in U | x \notin A\}$

(U is the universal set, containing all objects under consideration)

Symmetric difference: $A \Delta B = A \cup B - A \cap B$

Cartesian product: $A \times B = \{(x, y) | x \in A \text{ and } y \in B\}$

Power set of A = $\mathcal{P}(A) = \{X | X \subseteq A\}$
= Set of all subsets of A

Some properties of sets

- Two sets are EQUAL if every element of one is an element of the other
- The union and intersection operations satisfy associative, commutative, and distributive laws (just like addition and multiplication).
- Complement of complement of a set is the set itself
- Empty set is denoted as $\{ \}$ or \emptyset

Example 1

Let the universal set U be \mathbb{Q} , the set of rational numbers.

$$A = \left\{ x \in \mathbb{Q} \mid x = \frac{2m}{3n}, \gcd(2m, 3n) = 1 \right\}$$

($\gcd = 1 \implies$ no common factors between $2m, 3n$)

$$B = \left\{ x \in \mathbb{Q} \mid x = \frac{5m}{7n}, \gcd(5m, 7n) = 1 \right\}$$

Questions for the sets in Example 1

1. Describe the following:

$$A \cap B, A \cup B, A - B, \bar{A},$$
$$\overline{A \cap B}, \overline{A \cup B}, \overline{A} \cup \overline{B}, \overline{A} \cap \overline{B}.$$

2. What happens if we remove the condition that the numerator and denominator have no common factors?

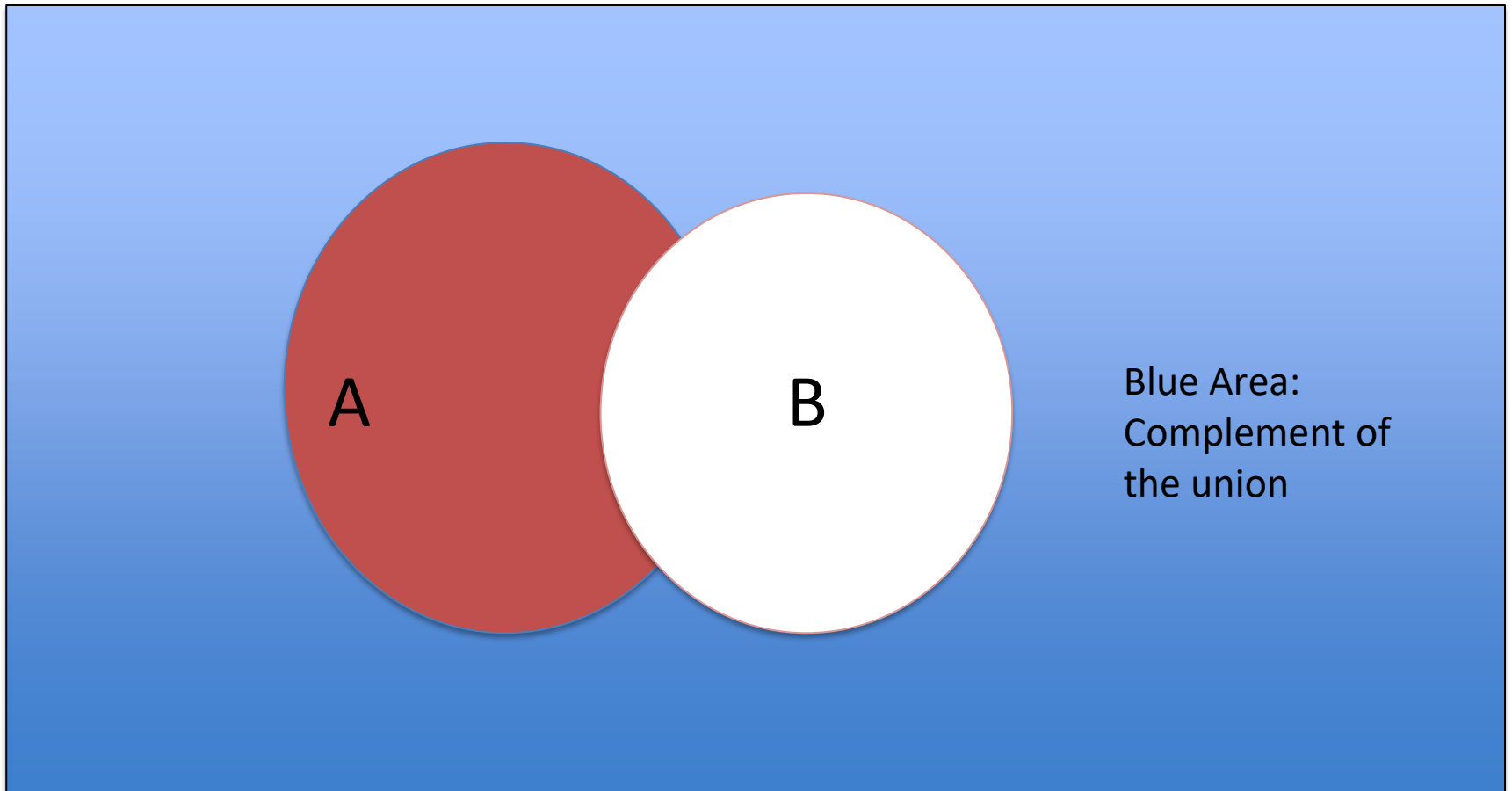
De Morgan's Law for sets

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$
$$\overline{A \cap B} = A \cup B$$

Complement of union equals intersection of complements

Complement of intersection equals union of complements

De Morgan's Law in pictures



Some exercises in set theory

1. The set of “all sets that are not members of themselves” in set builder notation: $\{X \mid X \notin X\}$
2. Check DeMorgan’s law if $A = \{1, 2, 3\}$,
 $B = \{2, 3, 4\}$ and the universal set is $\{1, 2, 3, 4, 5\}$.
3. Is $A \subseteq B$? Is $B \subseteq U$? Is $A = B$?
4. Write the complements of the sets A, B, C, D described earlier.
(A = People who know....)

Exercise on subsets

- The set of integers Z is a subset of the set of rational numbers Q .
- So $Z \cap Q = Z$ and $Q \cup Z = Q$.
- Moreover, the complement of Z contains the complement of Q .
- In other words, $U - Q$ is a subset of $U - Z$ where $U - Q$, $U - Z$ denote the complements of Q and Z .
- (U is the universal set. Here it could be R , the set of real numbers)

Using Venn diagram to solve set theory problems

1.1.66 A television poll of 151 persons found that 68 watched “Law and Disorder”; 61 watched “25”; 52 watched “The Tenors”; 16 watched both “Law and Disorder” and “25”; 25 watched both “Law and Disorder” and “The Tenors”; 19 watched both “25” and “The Tenors”; and 26 watched none of these shows. How many watched all three?