11-3-2025 Notes, Proofs and Problem Solving 1

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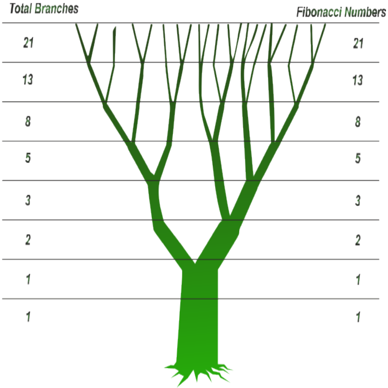
Outline

Fibonacci Sequence

1,1,2,3,5,8,13,...

Let a_n denote the $n-$th term for $n = 1, 2, 3, ...$
(You can think of a_n as a function $f : \mathbb{N} \to \mathbb{C}$).
$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ *for* $n \geq 3$.
Actually $a_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$ where $\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$.



Geometric Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, ...\}$

Answer: This is a **geometric** sequence.
Each term is *multiplied* by same number,
called common ratio and denoted r.
To get to $n-$th term you need to multiply $n-1$ times.
Here $r = -1/4$ and $a_n = 4r^{n-1}$.

Geometric sequences are the **discrete** analogs of the exponential function. (and vice versa).

Example 1 – Geometric sequence

Find a formula for the *n*-th term:

$$\frac{3}{5}, \frac{2}{3}, \frac{20}{27}, \frac{200}{243}, \ldots.$$

Answer for Example 1

Answer: Common ratio is $r = 10/9$. First term is 3/5.

To get to $n-$th term you need to multiply $n - 1$ times.

Here $r = -1/4$ and

$$a_n = \frac{3}{5} r^{n-1}.$$

Check by plugging in!

Arithmetic Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence

$$\{5, 8, 11, 14, 17, ...\}$$

This is an **arithmetic** sequence.
Each term is *increased* (or decreased) by same number, called common difference and denoted by d.
To get to $n-$th term you need to add $n-1$ times.
Here $d = 3$ and $a_n = 5 + 3(n-1) = 3n + 2$.

This is also same as formula for linear function. Linear function increases by the value of the slope each time x is increased by 1.

Example 2 – Arithmetic Sequence

$$1, 0, -1, -2, -3, ...$$

Write a formula for the n-th term.

The common difference is -1.

To get to $n-$th term you need to add $n - 1$ times.

Here $d = -1$ and $a_n = 1 + (-1)(n - 1) = 2 - n$.

Oscillating Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence

$$\{1, 0, -1, 0, 1, 0, -1, 0, \ldots\}$$

This in example of an **oscillating** sequence.
In fact you might recognize it as the values of the sine function at

$$\frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}, \ldots$$

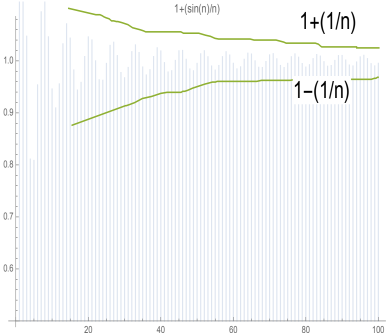
$$a_n = \sin \frac{\pi}{2} n.$$

You can also use the cosine function or do it using powers of -1.

Example for squeeze theorem

Sequences : part of a function. Graph of $b_n = 1 + (\sin(n)/n)$ at 1,2,3...,99,100 :

Each vertical line represents a value of the sequence.
The values of b_n are squeezed between $a_n = 1 + (1/n)$ and $c_n = 1 - (1/n)$.
By squeeze theorem (see below), since $a_n \to 1, b_n \to 1$, and $-1 \leq \sin(n) \leq 1 \implies a_n \leq b_n \leq c_n$, we get $b_n \to 1$.



Limit of a sequence – definition

Finding limits of sequences is similar to finding limits of functions.

$$\lim_{n\to\infty} a_n = L \textit{ if } \forall\epsilon \; \exists \; N \textit{ for which } \; |a_n - L| < \epsilon \, \forall \; n \geq N$$

If L exists the sequence is said to be **convergent.**

FACT: If the function $f(x) \to L$ as $x \to \infty$ then $f(n) \to L$ also, as $n \to \infty$

Proof: exercise

Limit of a sequence : Squeeze theorem

SQUEEZE THEOREM FOR SEQUENCES

If $a_n \leq b_n \leq c_n$ for $n \geq N$ and $a_n \to L, c_n \to L$, then $b_n \to L$ also.

Proof: $a_n \to L, c_n \to L$ means for any $\epsilon > 0$, $|a_n - L| < \epsilon$ for $n \geq N_1$ for some N_1 and $|c_n - L| < \epsilon$ for $n \geq N_2$ for some N_2. Choosing N to be the bigger of N_1, N_2, we have:

$$L - \epsilon < a_n < L + \epsilon, L - \epsilon < c_n < L + \epsilon, \forall n \geq N.$$

$$\implies L - \epsilon < a_n, c_n < L + \epsilon, \forall n \geq N.$$

But $a_n \leq b_n \leq c_n$ means $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon, \forall n \geq N$. Thus $L - \epsilon < b_n < L + \epsilon, \forall n \geq N$ for any $\epsilon > 0$ and limit as $n \to \infty$ of b_n is also L.

Limit of a sequence – boundedness

CONVERGENCE OF BOUNDED, MONOTONIC SEQUENCES

BOUNDED: terms of sequence are always above or below a certain value. Either $a_n \leq M$ or $a_n > m$ for all n.
MONOTONIC: Either increasing or decreasing.

Exercise 1

Write down a formula for the $n-$th term (a_n) of the sequence and find its limit, if it exists:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.$$

Is it bounded? If so what is the bound?
Is it monotonic?

Exercise 1

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Is it bounded? If so what is the bound?
Is it monotonic?

Exercise 1 – bounded

It is bounded because $a_n < 1$ for all n.

We know this because the denominators are always bigger than the numerators.

In fact 1 is the smallest upper bound. Anything bigger than 1 is also an upper bound. We will see in next slide that 1 is the limit.

Exercise 1 – monotonic

It is also monotonic increasing, i.e, always increasing.
Each term is smaller than the next.
$a_n = n/(n+1)$. Putting $n+1$ instead of n we get the next term
$a_{n+1} = (n+1)/(n+2)$.

$a_n < a_{n+1}$ because $n/(n+1) < (n+1)/(n+2)$.
To prove this cross multiply and check that $n(n+2) < (n+1)^2$.
Note: This can be done because all terms are positive.

Exercise 1 – using squeeze theorem

The **squeeze (sandwich) theorem** is really redundant here but you can still use it in the following way:

$$1 - \frac{1}{n} < \frac{n}{n+1} = 1 - \frac{1}{n+1} < \frac{n}{n} = 1$$

Both $1 - \dfrac{1}{n}$ on the left and 1 on the right go to 1,
so the sequence is squeezed or sandwiched between two sequences
both of which go to 1. So the limit of given sequence is also 1.

Exercise 1 – Using L'Hospital's rule

You can also use L'Hospital's because
both numerator and denominator go to ∞.

$$\lim_{x\to\infty}\frac{x}{x+1}=\lim_{x\to\infty}\frac{x'}{(x+1)'}=\lim_{x\to\infty}\frac{1}{1}=1.$$

Since the real valued function $\dfrac{x}{x+1}\to 1$ as $x\to\infty$,
the sequence $\dfrac{n}{n+1}\to 1$ as well.

Limit of a sequence – boundedness

THEOREM: EVERY BOUNDED, MONOTONIC SEQUENCE IS CONVERGENT.

Proof is based on the COMPLETENESS AXIOM for the set of real numbers:
Every non-empty, bounded above subset of $\mathbb{R}$ *has a least upper bound*

Limit of a sequence – boundedness – proof

PROOF: Assume that sequence is monotonic increasing. (Proof is similar if decreasing).
Suppose L is a least upper bound for a_n. Then for any $\epsilon > 0$ we must have $a_N > L - \epsilon$ for some N.
Since sequence is increasing, $a_n > a_N > L - \epsilon$ for all $n \geq N$.
Also $a_n < L$ because L is upper bound, so $L - \epsilon < a_n < L$ for all $n \geq N$.
Hence $|a_n - L| < \epsilon, \forall n \geq N$ and so $a_n \to L$ as $n \to \infty$.

Exercise 18.3 – a recursive sequence

18.3 (d) (Lay's book, chapter on bounded sequences).
Show that the sequence s_n defined by $s_{n+1} = \sqrt{2s_n + 1}, n \geq 1, s_1 = 2$ is bounded, increasing, and find its limit.

Solution:
Increasing: Prove this by induction.
$s_1 = 2$ is positive. $s_2 = \sqrt{2(2) + 1} = \sqrt{5} > 2 = s_1$. So the sequence is > 2 and increasing at first. Assume true for $n - 1$.
So $2 < s_{n-1} \leq s_n$ and from this $2s_{n-1} + 1 \leq 2s_n + 1$.
Since they are all positive, we can take square root of both sides. Moreover, square root function is increasing, so we get $\sqrt{2s_{n-1} + 1} \leq \sqrt{2s_n + 1}$. But this means $s_n \leq s_{n+1}$ and induction is complete.

Exercise 18.3 – boundedness

....Continued

Boundedness: (with input from Kevin N).
Since we proved that sequence is increasing, we have
$2 < s_n \leq s_{n+1} = \sqrt{2s_n + 1}$ for all n.
Moreover all terms are positive (we will use that below to keep the inequality unchanged while squaring both sides).
$\implies s_n^2 \leq 2s_n + 1 \implies s_n^2 - s_n - 1 \leq 0.$
Now by what we know about quadratic inequalities, we must have $1 - \sqrt{2} \leq s_n \leq 1 + \sqrt{2}$ for all n.
Note that $1 - \sqrt{2}, 1 + \sqrt{2}$ are roots of the equation.
So $s_n \leq 1 + \sqrt{2}$ for all n.

Exercise 18.3 – limit

(Continued)

Finding limit : For this first we need to prove $\lim s_{n+1} = \lim s_n$ as $n \to \infty$. Basically that $s_2, s_3, ...$ has the same limit as $s_1, s_2, ...$ In fact it is true for s_{n+k} for any positive integer k – what matters is the values “at infinity." We want $|s_n - L| < \epsilon$ for any ϵ as long as $n \geq N$. Just take N to be big enough that the first few values don’t matter. Proof given in class.

Now take limit of both sides of $s_{n+1}^2 = 2s_n + 1$. Using the theorem just proved, get $s^2 = 2s + 1$ where s is the limit of both s_{n+1} and s_n. Solving, we get $s = 1 + \sqrt{2}$. (Note: we also used some other facts about limits – see next page).

Some basic theorems on limits

In the proof of the limit of $s_{n+1} = \sqrt{2s_n + 1}$ we used several formulae that are satisfied by all limits. For example, Limit of square of s_n is square of limit of s_n, limit of $2s_n$ is 2 times limit of s_n, etc., Here is a list of some basic limit theorems. Proof is exercise.
Assume s_n, t_n are convergent sequences, with limits s, t respectively.

1. Every subsequence of s_n also converges.
2. $s_n \pm t_n \to s \pm t, s_n t_n \to st, s_n/t_n \to s/t(if\ t \neq 0\ )$.
3. $\lim ks_n = k(\lim s_n)$ with k constant.
4. $s_n \leq t_n \implies s \leq t$.

Some basic theorems on limits of functions ; Continuity

All functions are real valued functions of real numbers.

$$\lim_{x \to c} f(x) = L \iff \forall \epsilon > 0, \exists \delta \textit{ such that } |x - c| < \delta \implies |f(x) - L| < \epsilon.$$

1. Limit of sum, product and quotient of functions follow similar rules as for sequences.
2. Continuity: A function is continuous at c if
$$\lim_{x \to c} f(x) = f(c).$$
3. If a sequence of real numbers $x_n \to c$ and $f(x)$ is continuous, then the sequence $f(x_n)$ converges to $f(c)$.

Exercises on continuity and sequences

Find the limits of the following sequences using appropriate continuous functions.
For example, for $s_{n+1} = \sqrt{2s_n + 1}$ we could have used $f(x) = \sqrt{2x + 1}$.
In all cases $n \to \infty$. For $f(x) \to c$ as $x \to \infty$ show that for some K, $|f(x) - c| < \epsilon$ whenever $x > K$.
To show function goes to infinity, show that for any big positive real number M there is some K, such that $|f(x)| > M$ whenever $x > K$.

1. $\sin(1/n)$
2. $e^{\frac{n+1}{n}}$
3. $\sqrt{2^n}$

Examples of discontinuous functions

From exercises in chapter on continuous functions:
Examples of functions $f(x)$ that are (Prove each one !)
(a) discontinuous everywhere
(b) continuous at exactly one value of x
(c) discontinuous everywhere but f^2 is continuous

$$(a) \qquad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - Q \end{cases}$$

$$(b) \qquad f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} - Q \end{cases} \qquad \text{(continuous at 0)}$$

$$(c) \qquad f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} - Q \end{cases}$$

Density of Rationals in the Real Numbers

The examples of discontinous functions above are based on the fact that
In every interval of real numbers there are rational numbers as well as irrational numbers
Also every real number is the limit of a sequence of rational numbers – this follows from the decimal approximation. For example $\sqrt{2}$ is the limit of 1, 1.4, 1.414, ... We say that the set of rationals $\mathbb{Q}$ is **dense** in the set of reals $\mathbb{R}$.

Proof that every real number is the limit of a sequence of rational numbers – below.

Proof that every interval has rationals and irrationals

It is enough to show that every interval has irrational numbers in it. Since the rationals are dense in the real numbers, we can find rationals that are also within same interval.

Let $\epsilon > 0$ be as small as we want, very close to 0, say smaller than 0.0001.
To find an irrational number in $(0, \epsilon)$, just take any irrational like π and find an integer N such that $\pi/N < \epsilon$. Then π/N is irrational (why?) and in $(0, \epsilon)$.
Now to find an irrational in any interval (a, b) we do a translation from a small interval near 0. First find a rational number r such that $a < r < b$ and then find a very small $\epsilon < b - r$. Now find an irrational $s \in (0, \epsilon)$. Then $a < r + s < b$ and $r + s$ is irrational (why?).

Decimal approximation of real numbers

Proof that every real number is a limit of a sequence of rational numbers of the form $a_i/10^{k_i}$ where a_i is an integer and k_i is a non-negative integer.

Proof: Let r be any real number and let $N \leq r$ be the greatest integer $\leq r$. Now define $r - N = r_1$. Then $0 \leq r_1 < 1$. (Why can't $r - N$ equal 1?)
Now find a positive integer k_1 such that $10^{k_1} r_1 \geq 1$. Let N_1 be the greatest integer smaller than $10^{k_1} r_1$. Let $r_2 = 10^{k_1} r_1 - N_1$.
Then as before $0 \leq r_2 < 1$.
Now we have $r = N + r_1 = N + (N_1 + r_2)/10^{k_1}$.
Continued....

Decimal approximation of real numbers – page 2

(Continued) We had $r = N + r_1 = N + (N_1 + r_2)/10^{k_1}$.

Let $a_1 = 10^{k_1} N + N_1$. Then $r = \frac{a_1}{10^{k_1}} + \frac{r_2}{10^{k_1}}$. We have

$$\left| r - \frac{a_1}{10^{k_1}} \right| = \frac{r_2}{10^{k_1}} < \frac{1}{10^{k_1}} \text{ because } r_2 < 1.$$

Continuing this process by induction, we can get a_i and k_i such that

$$\left| r - \frac{a_i}{10^{k_i}} \right| < \frac{1}{10^{k_i}} < \epsilon \text{ for any } \epsilon > 0 \text{ for some } k_i.$$

Decimal approximation of $\sqrt{2}$

Demonstrate the proof above with approximation for $\sqrt{2}$. First note that $1 < \sqrt{2} < 2$. So $N = 1, r_1 = \sqrt{2} - 1$.
Then note that $10(\sqrt{2} - 1) \geq 1$ because $10\sqrt{2} \geq 11$ because $200 > 121$. So $k_1 = 1$ in the above proof and $11 \leq 10\sqrt{2} < 20$ because $10\sqrt{2} \geq 11$ and $\sqrt{2} < 2$.
This means for some $k \in \{1, 2, 3, ..., 8, 9\}$

$$10 + k \leq 10\sqrt{2} < 11 + k \implies (10 + k)^2 \leq 200 < (11 + k)^2$$

[You can divide $[10, 20]$ into 10 intervals and $10\sqrt{2}$ is in one of them].

(Continued...)

Decimal approximation of $\sqrt{2}$ – page 2

Continued from above. Need $k \in \{1, 2, 3, ..., 8, 9\}$ such that

$$10 + k \leq 10\sqrt{2} < 11 + k \implies (10 + k)^2 \leq 200 < (11 + k)^2$$

Basically k is smallest in $\{1, 2, .., 9\}$ such that
$(10 + k)^2 \leq 200 < (11 + k)^2$. This happens for $k = 4$:
$14^2 \leq 200 < 15^2$. So $14 \leq 10\sqrt{2} < 15$ or $1.4 \leq \sqrt{2} < 1.5$. This can be continued indefinitely.
In the proof above, $N_1 = 4, r_2 = 10\sqrt{2} - 14, a_1 = 14$.
Then $r_2 = 10\sqrt{2} - 14 < 1$ because $10\sqrt{2} < 15$ because $200 < 225$. So the proces can be continued.

Intermediate Value Theorem - Statement

Theorem

If $f(x) : [a, b] \to \mathbb{R}$ *is a continuous real valued function such that* $f(a) < 0, f(b) > 0$ *then there is a point* $c \in [a, b]$ *such that* $f(c) = 0.$ *[Proof in book]*

In plain English, this means that if a function has positive and negative values on a CLOSED interval then its graph crosses the x-axis somewhere in that interval.

The fact that interval is closed and hence *includes* endpoints a and b is important. Otherwise, it could be negative at a and just approach 0 as $x \to a$ but never equal 0.

Intermediate Value Theorem - Application

Theorem

If $f(x)$ is a polynomial function of odd degree then there is a point $c \in \mathbb{R}$ such that $f(c) = 0$.

It is proved in book that polynomial functions are continuous. So we just need to show that it has positive and negative values. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + ...a_1 x + a_0$ and n be odd.

$$f(x)/x^n = a_n + \frac{a_{n-1}}{x} ... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \rightarrow a_n \text{ as } |x| \rightarrow \infty.$$

(The fractional terms go to 0 because we can find x such that $|x^k| > |a_k/\epsilon|$ for any $\epsilon > 0,\ k = 0, 1, 2, 3, ..., n-1$ and so $|a_k/x^k| < \epsilon$.)

Intermediate Value Theorem - Application – contd

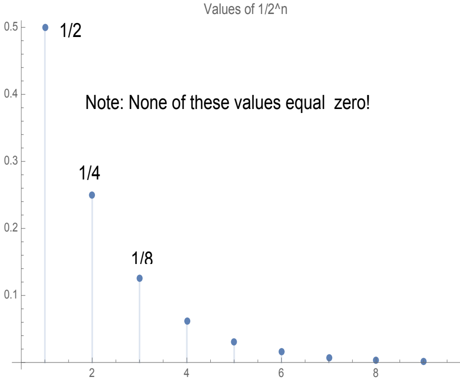
(Continued from above) We showed: $f(x)/x^n \to a_n$ as $|x| \to \infty$.

So now suppose $f(x)$ has always the same sign for all x. a_n is fixed, so it has the same sign too, for all x. If $f(x) = 0$ for some x we are done. So assume $f(x)$ is always positive or always negative.

Since $f(x)/x^n \to a_n$ as $n \to \infty$, there is an N such that $|x| \geq N$ means $f(x)/x^n$ is close to a_n and so $f(x)/x^n$ has the same sign as a_n. This means, whether $x > N$ or $x < -N$ (remember, we were looking at $|x| \to \infty$), $f(x)/x^n$ has the same sign. This is not possible because $f(x)$ has same sign everywhere and x^n is negative for large negative values and positive for large positive values, *if* n is odd. So $f(x)$ must change signs also.

Infinite Sums with Finite Value

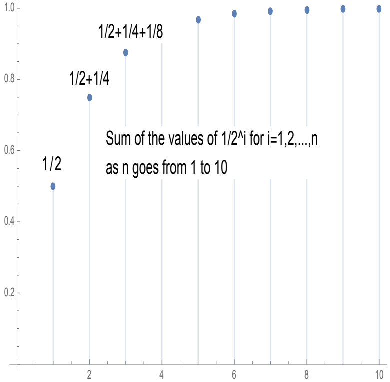
IS IT POSSIBLE TO ADD AN INFINITE SET OF NUMBERS AND GET A FINITE VALUE AS ANSWER?
ANSWER: YES!! Kind of happened with integrals: Gabriel's horn has finite volume.
YOU CAN THINK OF SUM ON RIGHT AS AN APPROXIMATION TO AN INTEGRAL!



Note: None of these values equal zero!

Infinite Sums with Finite Value – 2

SO WHAT IS THE SUM?
NOTICE THE PATTERN:
$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}=1-\frac{1}{4}$
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{3}{4}+\frac{1}{8}=\frac{7}{8}=1-\frac{1}{8}\ldots$
In general,
$\frac{1}{2}+\frac{1}{4}+\ldots.+\frac{1}{2^n}=1-\frac{1}{2^n}$
The reason this is happening is that *each time we are adding half the distance between the previous sum and 1.*



Geometric Sequence sum

FORMULA FOR SUM OF GEOMETRIC SEQUENCE

$$\begin{aligned} S(n) &= 1 + r + r^2 + \ldots r^n \\ \implies rS(n) &= r + r^2 + \ldots + r^{n+1} \\ \implies (1-r)S(n) &= 1 - r^{n+1} \\ \implies S(n) &= \frac{1}{1-r}\left(1 - r^{n+1}\right) \end{aligned}$$

THE ABOVE FORMULA IS VALID FOR ANY $r \neq 1$!!

Example 1 – Geometric sequence

Suppose you deposit \$ 5000 in an account each month for 10 years at a return of $i = 0.08$ (8 percent)
The amount from the first month results in $5000(1+i)^{10}$ after 10 months.
The amount from the second month results in $5000(1+i)^{9}$ after 9 months.
and so on .....
The last amount gives no return, so results in just 5000.
What is the total amount that you would have after 10 months?

Answer for Example 1

Use formula for geometric series :
$1 + r + r^2 + ... + r^{n-1} = \dfrac{1 - r^n}{1 - r}$.
Here we have $r = 1 + i = 1.08$.

$$
\begin{aligned}
& S(10) = 5000 + 5000r + 5000r^2 + ... + 5000r^{10} \\
\implies & S(10) = 5000(1 + r + ... + r^{10}) \\
\implies & S(10) = 5000 \times \frac{1 - r^{11}}{1 - r} \\
\implies & S(10) = 5000 \times \frac{1 - (1.08)^{11}}{1 - 0.08} \\
\implies & S(10) = 5000 \times \frac{1.3316}{0.08} = 83,227.44\$
\end{aligned}
$$

What is a series?

A SERIES IS A SEQUENCE OF SUMS

We call the limit of the series as the limit of the sum.

Note that the limit of a_n is not the same as limit of $\sum a_n$!

Infinite sum or geometric series sum

Using the formula

$$1 + r + r^2 + ... + r^{n-1} = \frac{1 - r^{n+1}}{1 - r}$$

$$1 + \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} = \frac{1}{1 - (1/2)}\left(1 - \frac{1}{2^{n+1}}\right)$$

$$= 2\left(1 - \frac{1}{2^{n+1}}\right) = 2 - \frac{1}{2^n}$$

$$\implies \frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} = 1 - \frac{1}{2^n} \to 1 \; as \; n \to \infty.$$

We say that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Question on series

Find an ϵ such that the geometric series sum s_n is within 0.0001 of 2.
Here s_n is the sum of the sequence 1, 1/2, 1/4, ...$1/2^n$...

Infinite geometric series – properties

Just as with integrals, $\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=1}^{n} a_k$.

We say the limit of $\sum_{k=1}^{n} a_k$ approaches L if $\left| \sum_{k=1}^{n} a_k - L \right| \to 0$ as $n \to \infty$. Or in more fancy words, for some N, the distance $\left| \sum_{k=1}^{n} a_k - L \right| < \epsilon$ for any $\epsilon > 0$ whenever $n > N$.

Here we have $\left| \sum_{k=1}^{n} a_k - L \right| = \left| \left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^n} \right) - 1 \right| =$
$\left| \left(1 - \frac{1}{2^n} \right) - 1 \right| = \frac{1}{2^n} \to 0$

Infinite geometric series – properties

In general, if $-1 < r < 1$ then the sum of the geometric series

$$1 + r + ... + r^n = \frac{1 - r^{n+1}}{1 - r} \to \frac{1}{1 - r}.$$

If $r > 1$ or $r < -1$ (i.e, $|r| > 1$) then the series diverges.
If the geometric sequence starts with a number a instead of 1, the sum equals

$$a+ar+...+ar^n = a(1+r+r^2+...+r^n) = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \to \frac{a}{1 - r}.$$

A telescoping series

The sum of the sequence $\dfrac{1}{k(k+1)} = \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{12}, \ldots$ is an example of a telescoping sum.

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right)$$

$$+ \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$$

So this sum is also finite, approaching 1.

Note that this is close to the sum of $1/k^2$.

Example 2: telescoping series

ANOTHER TELESCOPING SUM (11.2.48)

$$\sum_{k=2}^{\infty} \frac{1}{n^3 - n}$$

Note that the sum starts with a_2, not a_1.
[In fact, you can ignore first several terms when testing convergence and finding sum].
Start by factoring $n^3 - n$ to get $n(n^2 - 1) = n(n-1)(n+1)$.
Next use partial fraction decomposition.

Example 2: telescoping series – page 2

$$\frac{1}{n^3 - n} = \frac{-1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}.$$

You can try to write out this sum and cancel terms but it is easier if you modify it first.

Write

$$\frac{-1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)} = \frac{-1}{2n} + \frac{-1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}$$

and then rearrange it as $\frac{1}{2}\left(\frac{1}{(n-1)} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n}\right)$.

$$So \sum_{k=2}^{\infty} \frac{1}{n^3 - n} = \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{(n-1)} - \frac{1}{n}\right) + \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right)$$

Example 2: telescoping series – page 3

The two sums on the right are telescoping sums.

In fact you can show that the first is $\frac{1}{2}\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right)$

and that the second is $-\frac{1}{2}\left(\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right)\right)$

in exactly the same way as we evaluated the telescoping sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}.$$

Example 2: telescoping series – page 4

Finally

$$\sum_{k=2}^{\infty} \frac{1}{n^3 - n}$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right) - \frac{1}{2}\left(\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

If sum is finite, terms go to zero

You might have noticed that in the sum of geometric series, the terms get very small, in fact they are almost equal to zero, eventually.
Without this it would not be possible to add infinitely many and still get a finite sum.
This is stated as a theorem (proven fact) below:

Theorem: If $\displaystyle\sum_{k=1}^{\infty} a_k = L$, a finite quantity, then the $n-$th term $a_n \to 0$.

Proof: If sum is finite, terms go to zero

The following proof of this theorem is slightly different from book proof.
Here only to show you how you logically establish what I just mentioned above.

Proof: As we saw in previous page, $\sum_{k=1}^{\infty} a_k = L$ means that

$\left|\sum_{k=1}^{\infty} a_k - L\right| < \epsilon$ for any ϵ however small for all n eventually.

If $|(a_1 + a_2 + ... + a_{n-1} + a_n) - L| < \epsilon$ for all $n > N$,
then in particular it is true for both n and $n + 1$.

Proof: If sum is finite, terms go to zero – continued

So $|(a_1 + a_2 + ... + a_{n-1} + a_n) - L| < \epsilon$ and
$|(a_1 + a_2 + ... + a_{n-1} + a_n + a_{n+1}) - L| < \epsilon$.
Remember the triangle inequality $|A + B| \leq |A| + |B|$.
(If you haven't seen it just assume it true for the moment).
Let $S_n = a_1 + a_2 + ... + a_{n-1} + a_n$,
$A = S_{n+1} - L = a_1 + a_2 + ... + a_{n-1} + a_n + a_{n+1} - L, A + B = a_{n+1}$.
Then $B = L - (a_1 + a_2 + ... + a_{n-1} + a_n) = L - S_n$.

Proof: If sum is finite, terms go to zero – conclusion

Note that $|L - S_n| = |-(S_n - L)| = |S_n - L|$ because both positive and negative of same number have same absolute value.
By above conclusion we have $|S_n - L| < \epsilon$ AND $|S_{n+1} - L| < \epsilon$.
Now $|A + B| \leq |A| + |B| \implies |a_{n+1}| \leq |S_{n+1} - L| + |L - S_n| = |S_{n+1} - L| + |S_n - L| < \epsilon + \epsilon = 2\epsilon$.
So a_{n+1} can also get as small as we want.

Divergence Test

NOTE: The reverse doesn't work, though.
The terms can go to zero, but still the sum can be infinite!
As shown in book, $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ..$
goes to infinity.
You can kind of see it using the integral of $1/x$.
But if a_n does not go to zero, then *the series is guaranteed to diverge.*
This is called the **divergence test.**

Divergence Test – example

WARNING: THE SUM OF THE SERIES AND
THE LIMIT OF THE TERMS OF THE SEQUENCES
ARE TWO DIFFERENT THINGS.

The sequence $\{n/(n+1)\} \to 1$ i.e, $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, ... \to 1$ but the sum $1 + \frac{1}{2} + \frac{2}{3} + ... + \frac{n}{n+1} + ..$ does not go to 1.
In fact by the divergence test the sum is not finite because the $n-$th term $a_n = n/(n+1) \to 1$ and not to 0.
We can actually prove it goes to ∞ because the terms all are about 1 eventually and so you are just adding 1 repeatedly.

EXAMPLE 3: DIVERGENCE TEST

$$\sum_{n=1}^{\infty} e^{1/n}$$

This diverges by the divergence test.

Since $1/n \to 0$ as $n \to \infty$, we get that $e^{1/n} \to e^0 = 1$ using the continuity of e^x.

Thus a_n does not go to zero and the series diverges.

Oscillating Series

A SEQUENCE THAT HAS DIVERGENT SUM BUT NOT INFINITE SUM

Consider $1, 0, -1, 0, 1, 0, -1, 0, 1, 0, ...$
This sequence oscillates and hence never goes to 0.
(In fact it is $a_n = \sin(n\pi/2)$ as seen in class).
You can see that the sum goes like $1, 1, 0, 0, 1, 1, 0, 0, ...$
This does not have a limit because the distance of this sequence of sums from any fixed number never goes to zero. In other words, it doesn't stay close to any one particular number, no matter how many terms of $1, 0, -1, 0, ...$ you add up. The sums are always alternating between 1 and 0 instead of staying close to one of the two.

Example 4: Geometric Series

ANOTHER GEOMETRIC SEQUENCE

Does the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ converge?
If so, what is the sum?

Example 4: Geometric Series – answer

$n-$th term $\dfrac{(-1)^n}{3^n} \to 0.$

This can be seen by using the squeeze or sandwich theorem:

$$\frac{-1}{3^n} \leq \frac{(-1)^n}{3^n} \leq \frac{1}{3^n}.$$

The sequences on the left and right both go to 0 so the given sequence will also go to 0.

You can also say that this is a geometric sequence with $r = -1/3$.

Since $-1 < r < 1$ the sequence will go to zero.

Since the $n-$th term goes to zero, the series *may* converge.

Example 4: Geometric Series – continued

In this case, we can show not only that it does converge, but also find the sum because it is a geometric series.

The sum goes as $\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \ldots$ with the first term being $a = \frac{-1}{3}$ and the common ratio being the same, i.e, $r = \frac{-1}{3}$.

Since $|r| = 1/3 < 1$, the series converges and the sum is given by

$$\frac{a}{1-r} = \frac{-1/3}{1-(-1/3)} = \frac{-1/3}{4/3} = \frac{-1}{3} \times \frac{3}{4} = \frac{-1}{4}.$$