Howard University Math Department

Final Exam Solutions

Instructions: **NO CALCULATORS OR CELLPHONES**

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 120 minutes; Total 200 points. Each 20 points.

Please read the questions carefully before answering.

1. Prove directly that every natural number a that is not divisible by 3 is of the form $3n-1$ or $3n+1$ for some positive integer n.

   Then use it to prove by cases that if a is not divisible by 3 then a^2-1 is divisible by 3.

   Solution: If a is not divisible by 3 then either a is 1 less than a multiple of 3 or 1 more than a multiple of 3. So it is of the form $3n-1$ or $3n+1$ for some positive integer n. (If it were of the form $3n+2$, i.e, 2 more than a multiple of 3, say like 2, 5,8, etc., then it is also of form $3n-1$ because $2 = 3-1, 5 = 6-1, 8 = 9-1$, etc.,)

   Prove by cases that in such a case a^2-1 is divisible by 3.

   $$a = 3n \pm 1 \implies a^2 - 1 = (3n \pm 1)^2 - 1 = (9n^2 \pm 6n + 1) - 1 = 9n^2 \pm 6n = 3 \times (3n^2 \pm 2n).$$

   So in both cases it is 3 times some natural number.

   This can also be done by factoring a^2-1 as $(a-1)(a+1)$ and then noting that either $a-1$ or $a+1$ must be divisible by 3.

2. We proved in class that in any group of six people there will be 3 who know each other or 3 who do not know each other. So $R(3,3) \leq 6$. Show that $R(3,3) = 6$ by proving using counterexamples that the same will not be true for any group of 5 people, 4 people, or 3 people. [It is trivially true for a group of 2 people or 1 person].

   Solution: The pentagon is a counterexample for groups of 5 people. Suppose there are five people A,B, C, D, E where A knows B, B knows C, etc., until you get E knows A and there are no other relationships. It has no triangle, meaning no three know each other. It also has no set of three who don't know each other because every person (point) knows two others (adjacent points). The proof for 4 and 3 are easy.

3. Write the negative, converse and contrapositive of the following statement. Then prove it if it is true, give a counterexample if it is false: "If a function maps the set X to the set Y in a one to one fashion, then every element in Y has a pre-image in X."

   Note: one-one function maps different x-values in X to different y-values in Y.

   Solution:

Negative (If A then NOT B) : If a function maps the set X to the set Y in a one to one fashion, then some element in Y does not have a pre-image in X.

Converse (If B then A) : If every element in Y has a pre-image in X then the function maps the set X to the set Y in a one to one fashion.

Contrapositive (If NOT B then NOT A) : If some element in Y doesn't have a pre-image in X then the function does not map the set X to the set Y in a one to one fashion.

The given statement is false. A very simple counterexample: Let X be the singleton set with only the element a. Let Y be the set with the elements b and c. We can make a one-one function by taking a to b or a to c but then either c or b won't have a pre-image.

A nice counterexample: $f(x) = e^x$ maps all real numbers to all real numbers but images are always positive so pre-images do not exist for negative real numbers

4. Prove by induction that 7 divides $n^7 - n$ for any natural number n. You may need to use the expansion of $(n+1)^7$ using binomial formula.

   For $n = 1$ we have $1^7 - 1 = 0$ and certainly 7 divides 0.

   Assume 7 divides $n^7 - n$. Then $n^7 - n = 7m$ for some integer m.

   Need to prove that 7 divides $(n+1)^7 - (n+1)$. In otherwords, $(n+1)^7 - n - 1 = 7m$ for some integer m.

   Using binomial formula, $(n+1)^7 = n^7 + 7n^6 + +\binom{7}{2}n^5 + \binom{7}{3}n^4 + \binom{7}{3}n^3 + \binom{7}{2}n^2 + 7n + 1$.

   Notice that we have used the symmetry of the binomial formula (or of Pascal's triangle).

$$(n+1)^7 - n - 1 = (n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1) - n - 1$$

$$= n^7 + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n + 1 - n - 1$$

   After cancelling 1 and rearranging this gives

$$(n+1)^7 - n - 1 = (n^7 - n) + 7n^6 + 21n^5 + 35n^4 + 35n^3 + 21n^2 + 7n = 7m + 7(n^6 + 3n^5 + 5n^4 + 5n^3 + 3n^2 + n) = 7k$$

   where k is what you get after taking out 7 and combining everything.

5. Use proof by cases to show that $3m + 5n^2 = 40$ has no solution in positive integers.

   Solution :

   First note that $n < 3$. Reason: If $n \geq 3$ then we get $3m = 40 - 5n^2 < 0$. But $3m$ is positive.

   Look at two cases $n = 1, 2$.

   If $n = 1$ we get $3m + 5 = 40 \implies 3m = 35$, and this has no solutions because 3 is not a factor of 35.

   If $n = 2$ we get $3m + 5(2)^2 = 3m + 20 = 40 \implies 3m = 20$. No solutions again because 3 doesn't divide 20.

   So there are no positive integer solutions to this equation.

   You can also do this by looking at $m = 1, 2, 3, ..13$ but that will be longer.

6. Prove by contradiction that $\sqrt{3}$ is irrational.

   Solution:

   Similar to the proof for $\sqrt{2}$ which was done in class.

7. Write the formula for the n-th term a_n of the following sequence in terms of n : $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, ...$.

   Prove **by induction** that $a_1 \times a_2 \times a_3 \times ... \times a_{n-1} \times a_n = 1/(n+1)$.

   Solution: $a_n = \dfrac{n}{n+1}$ is the formula. The equation works for $n = 1$. To go from n to $n+1$, we need to show that if $P_n = a_1 \times a_2 \times a_3 \times ... \times a_{n-1} \times a_n = 1/(n+1)$ then $P_{n+1} = a_1 \times a_2 \times a_3 \times ... \times a_{n-1} \times a_n \times a_{n+1} = 1/((n+1)+1) = 1/(n+2)$.

   To see this note that P_{n+1} contains the P_n. In fact, $P_{n+1} = P_n \times a_{n+1}$ and so

$$P_{n+1} = \frac{1}{n+1} \times a_{n+1} = \frac{1}{n+1} \times \frac{n+1}{n+2} = \frac{1}{n+2}.$$

8. Prove that $n! \geq 2^{n-1}$ is always true by using induction.

   Solution:

   Again, start with $n = 1$: We have $1! = 1 \geq 2^{1-1} = 2^0 = 1$.

   Next we need to show that we can go from any k to $k+1$.

   In other words, if $k! \geq 2^{k-1}$ for any of $1, 2, 3, ...$ up to infinity, then $(k+1)! \geq 2^{(k+1)-1} = 2^k$ also.

   This is the key step.

   To go from $k!$ to $(k+1)!$ first we note that $(k+1)! = (k+1)(k!)$.

   If we want to use that $k! \geq 2^{k-1}$ to prove $(k+1)! \geq 2^{(k+1)-1} = 2^k$ then we need to go from $k!$ to $(k+1)!$. Now in order to do this without changing the inequality $k! \geq 2^{k-1}$ we multiply BOTH sides by $k+1$.

$$k! \geq 2^{k-1} \implies (k+1)k! \geq (k+1)2^{k-1} \implies (k+1)! \geq (k+1)2^{k-1}.$$

   Now it is true that $k+1 \geq 2$ for all $k = 1, 2, 3, ...$ So

$$(k+1)! \geq (k+1)2^{k-1} \implies (k+1)! \geq 2(2^{k-1}) = 2^k$$

   Thus we have proved that, IF $k! \geq 2^{k-1}$ THEN $(k+1)! \geq 2^k$.

9. Prove by contradiction: if 35 coins are distributed among eight bags such that each bag contains at least one coin, then at least two bags contain the same number of coins.

   Solution: Assume all bags contain different number of coins, with each having at least one. The smallest number of total coins that the bags can contain is $1+2+3+...+7+8 = (8 \times 9)/2 = 36$ which is bigger than 35. So at least two bags should have same number of coins.

10. A triangular number is a number that is the area of an equilateral triangle. For example, the numbers in Pascal's triangle form an equilateral triangle. The number of numbers in a Pascal's triangle goes like 1, 1+2, 1+2+3, ...and so on. So the first 3 triangular numbers are 1, 3, 6. What would be the 100th triangular number?

    Similarly the pentagonal numbers count the number of dots arranged in a pentagonal shape, with the first number by convention being 1. The first few pentagonal numbers are 1,5,12,22,35,... What would be the 100th pentagonal number?

    Solution: The 100th triangular number is just the sum 1+2+3+...+99+100 whose sum is given using the formula for the sum of arithmetic sequences: Multiply the average of the first and last numbers by number of terms. Here it will be $100 \times (1 + 100)/2 = 50 \times 101 = 5050$.

    The pentagonal numbers are also sums of the terms of an arithmetic sequences: 1, 1+4, 1+4+7, 1+4+7+10,.... The 100th number in the arithmetic sequence 1,4,7,...is $1+(99 \times 3) = 1+297 = 298$. So the 100th pentagonal number is $1+4+7+...+295+298 = 100 \times (1+298)/2 = 50 \times 299 = 14950$.