9/30/2026 Fall 2026, Discrete Structures Test 1 Solutions Sitaraman

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1. (20 points) State the converse (4 points) , contrapositive (6 points) and negative (6 points) of the following statement about sets. (4 points) Is the converse true?

   If a set of integers is finite then it has a maximum element.

   Converse: If a set of integers has a maximum element then it is finite.

   It is not true. The set of negative integers is a counterexample.

   Contrapositive: If a set of integers doesn't have a maximum element then it is not finite.

   Negative: A set of integers is finite and it has no maximum element.

2. (20 points) Write the following statement and its negative in symbols. Denote the set of natural numbers by $\mathbb{N}$. You should write ALL of it in symbols.

   If A is a subset of the natural numbers then it has a least element (least element means something smaller than all the other elements).

   Solution:

$$A \subseteq \mathbb{N} \implies \exists x \in A \ni \forall y \in A, x \leq y.$$

$$(A \subseteq \mathbb{N}) \wedge (\nexists x \in A \ni \forall y \in A, x \leq y).$$

3. (20 points)

   a. Write the contrapositive and prove it:

      If $xy > 100$ then either x or y is greater than 10.

   b. Prove using definition of subsets:

      If X, Y are any two sets, then $X \subseteq Y \iff \overline{Y} \subseteq \overline{X}$.

   [If X is contained in Y then complement of Y is contained in complement of X].

   Solution:

   Proof:

   a. Contrapositive: If all of x, y are less than or equal to 10, then their product is also less than or equal to 100. You can see it by multiplying.

   2. $X \subseteq Y$ means $a \in X \implies a \in Y$. Contrapositive of that is $a \notin Y \implies a \notin X$ and this is same as $\overline{Y} \subseteq \overline{X}$. Show that argument works both ways.

4. (20 points) Prove by cases: $2m^2 + 3n^2 = 40$ has no solutions in positive integers.

   It is enough to check when $n = 1, 2, 3$ because if $n \geq 4$ then $3n^2 \geq 48$ and so $2m^2+3n^2 > 40$ for any positive integer m.

   (You can also check for $m = 1, 2, 3, 4$. Proof is similar).

   When $n = 1$, $2m^2 + 3(1^2) = 40 \implies m^2 = 37/2$ which is impossible for any positive integer m.

   When $n = 2$, $2m^2 + 3(2^2) = 40 \implies m^2 = 14$ which is impossible for any positive integer m.

   When $n = 3$, $2m^2 + 3(3^2) = 40 \implies m^2 = 13/2$ which is impossible for any positive integer m.

   So in all cases there is no solution.

5. (20 points) Show using regular induction that for any natural number n, the following is true:

$$\frac{1}{2n} \leq \frac{1 \times 3 \times 5 \times ..... \times 2n-1}{2 \times 4 \times 6 \times .... \times 2n}. \qquad (IH)$$

   Solution:

   Base case: Check this is true for $n = 1 : 1/2 \leq 1/2$ is true.

   Now assume true for n.

   It is important to keep in mind that the product contains ALL the odd numbers from 1 to $2n-1$ in the numerator and ALL the even numbers from 2 to $2n$ in the denominator

   Need to prove

$$\frac{1}{2(n+1)} \leq \frac{1 \times 3 \times 5 \times ..... \times (2(n+1)-1)}{2 \times 4 \times 6 \times .... \times 2(n+1)}.$$

   Note that the RHS contains the expression for n :

$$\frac{1}{2(n+1)} \leq \frac{1 \times 3 \times 5 \times ..... \times (2n-1) \times (2n+1)}{2 \times 4 \times 6 \times ....(2n) \times (2n+2)}. \qquad (1)$$

   Method 1

   The part for n is $\geq 1/2n$ by induction hypothesis. So multiplying the inequality from the induction hypothesis (IH), aka the inequality for n, on both sides by $(2n+1)/(2n+2)$ we get :

$$\frac{1}{2n} \times \frac{2n+1}{2n+2} \leq \frac{1 \times 3 \times 5 \times ..... \times (2n-1) \times (2n+1)}{2 \times 4 \times 6 \times ....(2n) \times (2n+2)}. \qquad (2)$$

The final destination is inequality (1). Call it $C \leq A$. The intermediate step is inequality (2). Call it $B \leq A$. If we can show that $C \leq B$ then in combination with $B \leq A$ we get $C \leq A$. Here is the proof that $C \leq B$: (proof is backwards)

$$\frac{1}{2n+2} \leq \frac{1}{2n} \times \frac{2n+1}{2n+2} \iff (2n)(2n+2) \leq (2n+1)(2n+2) \iff 4n^2+4n \leq 4n^2+6n+2$$

After cancellations the last inequality is $0 \leq 2n+2$. Since $n > 1$ we have $2n+2 \geq 0$ and we are done.

Method 2

Canceling $1/(2(n+1)) = 1/(2n+2)$ on both sides of inequality (1), and remembering that in the RHS, we have the numerator and denominator from step n contained in it, we get

$$1 \leq \frac{1 \times 3 \times 5 \times ..... \times (2n-1) \times (2n+1)}{2 \times 4 \times 6 \times .... \times 2n}.$$

Dividing both sides by $2n+1$ we get

$$\frac{1}{2n+1} \leq \frac{1 \times 3 \times 5 \times ..... \times 2n-1}{2 \times 4 \times 6 \times .... \times 2n}.$$

By induction hypothesis, $1/2n$ is smaller than or equal to product on RHS. But $1/(2n+1)$ is smaller than $1/2n$. So $1/(2n+1)$ is also smaller than RHS.

Thus we have proved the statement for $n+1$ and the proof is complete.

6. (10 points extra credit) Show using strong induction that for any natural number $n \geq 4$, the Fibonacci numbers defined by $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, ..., F_n = F_{n-1} + F_{n-2}$ satisfy

$$F_n < \left(\frac{7}{4}\right)^{n-2}.$$

Note that it *doesn't* work for n up to 3. For example, $(7/4)^{3-2} = 1.75$ but $F_3 = 2$.

Solution:

Check this is true for $n = 4$ and 5.

This is because we will need *two previous steps* to prove the n-th step.

Since by strong induction we assume the statement is true for all $k < n$, it could certainly be assumed true for $n-1$ and $n-2$.

So we will assume that $F_{n-1} < (7/4)^{n-3}$ and $F_{n-2} < (7/4)^{n-4}$ and try to prove it for n.

Combining the two inequalities we get

$$F_n = F_{n-1} + F_{n-2} < \left(\frac{7}{4}\right)^{n-3} + \left(\frac{7}{4}\right)^{n-4} = \left(\frac{7}{4}\right)^{n-4} \left(\frac{7}{4} + 1\right)$$

Now $(7/4) + 1 = 11/4$ and $(11/4) < (7/4)^2 = 49/16$ because $11/4 = 44/16$ (multiply above and below by 3).

Therefore in the last inequality would still hold if we replace $7/4$ by $(7/4)^2$.

We get

$$F_n < \left(\frac{7}{4}\right)^{n-4}\left(\frac{7}{4}+1\right) < \left(\frac{7}{4}\right)^{n-4}\left(\frac{7}{4}\right)^2 = \left(\frac{7}{4}\right)^{n-2}$$

Thus we have proved the statement for n and the proof is complete.