Howard University Math Department

1. (15 points) This question is about the string of 0's and 1's in an 8 digit binary number, such as 10101010. For each of the following, you must explain your answer step by step.

   (a) How many of the numbers will have exactly three ones?

   (b) How many of the numbers will have at least two zeroes?

   (c) How many of the numbers will have at most one zero?

   Solution:

   a) Here you just need to pick the 3 digits that have ones in them. The rest will be zero. Order of the digits doesn't matter because they are all ones. This can be done in $8C_3 = 8!/(5! \times 3!) = (8 \times 7 \times 6)/(3 \times 2 \times 1) = 56$.

   (b) At least two zeroes means the number of ways of getting exactly two zeroes OR exactly 3 zeroes, ..., OR exactly 8 zeroes. Easier to calculate this as the opposite of exactly one or no zeroes and subtract it from the total number of 8-digit binary numbers. There are totally $2^8 = 256$ binary numbers with 8 digits because each digit has two choices namely 0 and 1 and they can be repeated. It could be all 1's or all 0's, for example.

   Exactly 8 of them have only one digit as one – you could have first digit as 1, or second digit as 1 and so on. You can also say it is $8C_1 = 8$. Exactly one has no ones (all zeroes). This is also $8C_0 = 1$. So the number of digits with at least 2 zeroes is $2^8 - (8 + 1) = 256 - 9 = 247$.

   (c) This is the total of exactly one or no zeroes and this is $8 + 1 = 9$ as calculated in (b) above.

2. (a) (7 points) How many ways can you distribute 20 identical books in 3 different shelves?

   (b) (8 points) Same question as (a), except now instead of 20 identical books you have 8 copies of a math book, 5 copies of a physics book and 7 copies of a chemistry book in the same shelf.

   Solution:

   (a) This is generalized combinations: We are just dividing 20 books into 3 partitions. Let nC_k be denoted as $C(n, k)$. Answer is $C(20+3-1, 3-1) = C(22, 2) = 22!/(20!2!) = (22 \times 21)/2 = 231$. NOTE: $C(22, 2) = C(22, 20)$. In general, $C(n, k) = C(n, n - k)$.

(b) This is generalized permutation. You divide number of ways to arrange all 20 (as if they were different) and divide by number of ways to permute the math books, the physics books and chemistry books among themselves. If N is the answer,

$$N = \frac{20!}{8! \times 7! \times 5!}.$$

3. (Extra credit 10 points) How many ways to distribute 20 different books in 3 different shelves?

   Solution:

   This is just 3^{20} because for each book there are 3 choices of shelves and the shelf choices can be repeated. Like tossing a die with 3 faces 20 times.