Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 30 minutes

Please read the questions carefully before answering

1. (a) (9 points) Show that the following is an equivalence relation on the rational numbers :

$$xRy \iff x - y \in \mathbb{Z}.$$

(b) (6 points) Find the equivalence classes of 0 and 1.5.

Easy to check that this is an equivalence relation:

Reflexive: $\forall x \in \mathbb{Q}, xRx$ because $x - x = 0 \in \mathbb{Z}$.

Symmetric: $\forall x, y \in \mathbb{Q}, xRy \implies yRx : x - y \in \mathbb{Z} \implies y - x = -(x - y) \in \mathbb{Z}$ is true because negative of an integer is also an integer.

Transitive: $\forall x, y, z \in \mathbb{Q}, xRy \text{ and } yRz \implies xRz : x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z} \implies x - z = (x - y) + (y - z) \in \mathbb{Z}$ is true because sum of two integers is also an integer.

Equivalence class of 0 is $(0) = \{x \in \mathbb{Q} \mid xR0\} = \{x \in \mathbb{Q} \mid x - 0 \in \mathbb{Z}\} = \{x \in \mathbb{Z}\}$ is all the integers.

Equivalence class of 1.5 is $(1.5) = \{x \in \mathbb{Q} \mid xR1.5\} = \{x \in \mathbb{Q} \mid x - 1.5 \in \mathbb{Z}\} = \{x \in \mathbb{Q} \mid x = 1.5 + k, k \in \mathbb{Z}\} = \{...., -2.5, -1.5, 0, 1.5, 2.5, ...\}$.

2. (15 points) Given the sequence $s_n = \frac{1}{n} - \frac{1}{n+1}....., n \geq 1$ answer the following:

(a) Is it increasing, decreasing, nonincreasing or nondecreasing? (could be more than one).

NOTE: You MUST prove for the full sequence in general. Not enough to just check the first few terms.

(b) Find $\sum_{n=1}^{4} s_n$

(c) Find the formula for the k-th term of the subsequence $s_2, s_4, s_6, ...$ (with even indices only) considered as a new sequence $a_1, a_2, a_3, ...$

Solution:

a) We look at the difference between s_n and s_{n+1}. We get

$$s_n - s_{n+1} = \left(\frac{1}{n} - \frac{1}{n+1}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{n+1}\left(\frac{1}{n} - \frac{1}{n+2}\right) = \frac{1}{n+1} \times \frac{2}{n(n+2)}$$

The difference is always positive when $n \geq 1$. . So $s_n - s_{n+1} > 0$ always. So $s_n > s_{n+1}$ always and the sequence is decreasing.

Alternately You can also argue the following way:

$$s_n - s_{n+1} > 0 \implies \frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} > 0 \implies \frac{1}{n(n+1)} > \frac{1}{(n+1)(n+2)}$$

In the above we are using the simplification of $(1/n) - (1/(n+1)$ we got in the first proof. Now canceling $1/(n+1)$ from both sides we need $1/n > 1/(n+2)$ which is true because $n + 2 > n$.

b) Use the fact that

$$s_n = \left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{4} s_n = s_1 + s_2 + s_3 + s_4 = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{30+10+5+3}{60} = \frac{48}{60} = \frac{4}{5}.$$

c) We have $n = 2k$ and plugging it into $s_n = 1/n(n+1)$ we get $a_k = s_{2k} = 1/(2k(2k+1))$.

Check: $a_1 = s_2 = 1/6 = 1/(2 \times 3), a_2 = s_4 = 1/20 = 1/(4 \times 5), ...$

3. (Extra credit) (5 points) Can you see a pattern in the sums of the sequence in problem 2? What would be the sum of the first 100 terms, in other words, what is $\sum_{n=1}^{100} s_n$?

$s_1 + s_2 = 1/2 \ + \ 1/6 = 2/3$, but you can also see it this way:

$$s_1 + s_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = 1 - \frac{1}{3} = \frac{2}{3}.$$

You see that the middle term cancels and you are left with 1 minus the last term. Earlier we saw that sum of first 4 terms is 4/5 which is $1 - (1/5)$. Using the same pattern,

$$s_1 + s_2 + ... + s_{n-1} + s_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

So sum of first 100 terms is 100/101.