

Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 30 minutes

Please read the questions carefully before answering

1. (10 points) Prove by cases: $m^3 + n^2 = 40$ has no solutions in positive integers.

It is enough to check when $m = 1, 2, 3$ because if $m \geq 4$ then $m^3 \geq 64$.

When $m = 1$, $1^3 + n^2 = 40 \implies n^2 = 39$ which is impossible for any positive integer n .

When $m = 2$, $2^3 + n^2 = 40 \implies n^2 = 32$ which is impossible for any positive integer n .

When $m = 3$, $3^3 + n^2 = 40 \implies n^2 = 13$ which is impossible for any positive integer n .

2. (10 points) Prove by cases: For any two real numbers x, y ,

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}.$$

The three cases are: $x = y, x > y, x < y$.

When $x = y$, we have $\max\{x, y\} = x = y$. and $RHS = \frac{x+y+|x-y|}{2} = (x+x)/2 = x$ (you can also say it equals $(y+y)/2 = y$ because both are equal anyway).

When $x > y$, $\max\{x, y\} = x$ and $RHS = \frac{x+y+|x-y|}{2} = (x+y+x-y)/2 = x$ because $|x-y| = x-y$ when $x > y$.

When $y > x$, $\max\{x, y\} = y$ and $RHS = \frac{x+y+|x-y|}{2} = (x+y+y-x)/2 = y$ because $|x-y| = y-x$ when $y > x$.

Notice how the arguments are similar. So we really needed to do only one of $x > y$ or $x < y$.

3. (10 points) Prove **using induction** : If $n \in \mathbb{Z}$ then $n^3 + n$ is always even.

In the proof you can assume that $n^2 + n$ is always even, as we proved in class, if necessary.

Base case: $n = 1 : 1^3 + 1 = 2$ is even.

Induction hypothesis: Assume $n^3 + n$ is even.

Then we need to show $(n + 1)^3 + (n + 1)$ is even USING the induction hypothesis.

$$(n+1)^3 + (n+1) = n^3 + 3n^2 + 3n + 1 + n + 1 = n^3 + n + (3n^2 + 3n) + 2 = n^3 + n + 3(n^2 + n) + 2$$

All three terms of the final sum are even. The first one through induction hypothesis, the second one as mentioned in question, and the third one is just 2.

Alternately, you can try to write $n^3 + n = n(n^2 + 1)$ and try to prove that $(n + 1)^3 + (n + 1) = (n + 1)((n + 1)^2 + 1) = (n + 1)(n^2 + 1 + (2n + 1))$ is even but it takes a bit more work.

4. (10 points) Prove **using induction** :

$$1 + 4 + 7 + \dots + (3n - 5) + (3n - 2) = n(3n - 1)/2.$$

Base case: $n = 1$: The sum is 1 and the formula gives $1(3 - 1)/2 = 1$.

Induction hypothesis: Assume $1 + 4 + 7 + \dots + (3n - 5) + (3n - 2) = n(3n - 1)/2$.

Then we need to show USING the induction hypothesis :

$$\begin{aligned} 1 + 4 + 7 + \dots + (3n - 5) + (3n - 2) + (3(n + 1) - 2) &= (n + 1)(3(n + 1) - 1)/2 \\ &= (n + 1)(3n + 2)/2 \end{aligned}$$

$$\begin{aligned} &1 + 4 + 7 + \dots + (3n - 5) + (3n - 2) + (3(n + 1) - 2) \\ &= [1 + 4 + 7 + \dots + (3n - 5) + (3n - 2)] + 3n + 1 = (n(3n - 1)/2) + 3n + 1. \end{aligned}$$

Notice how we used the induction hypothesis. *We replaced the sum upto $3n - 2$ using the formula given in induction hypothesis.*

The key is to include the sum of n terms in the sum of the $n + 1$ terms.

Simplifying the RHS,

$$\frac{n(3n - 1)}{2} + 3n + 1 = \frac{3n^2 - n + 6n + 2}{2} = (3n^2 + 5n + 2)/2 = (n + 1)(3n + 2)/2$$

as required.