DISCRETE STRUCTURES 10/23/26 CLASS NOTES

RELATIONS

FACT: Equivalence relations divide a set into disjoint equivalence classes and vice versa.

A partition of a set into disjoint subsets automatically gives an equivalence relation.

Example A: suppose we divide HU students based on their majors, and no one has a double major, then "x has the same major as y" gives an equivalence relation on the set of students of HU.
The equivalence classes are just students with the same major.

Example B: If we divide real numbers into a disjoint union as
…. [-2,-1) U [-1, 0) U [0,1) U [1,2) then the relation is xRy iff Floor[x] = Floor[y].
(Floor function maps a real number to greatest integer smaller than that number).

Partial Order

A relation is a partial order if it is reflexive, antisymmetric, and transitive.

Antisymmetric means xRy and yRx both true iff x = y.

Example 1: (3.3, 29) Check if the relation on positive integers given by $xRy\ if\ x \geq y$ is reflexive, symmetric, antisymmetric and transitive.

Reflexive: Is any x related to x itself? Yes. x = x, so it satisfies $x \geq x$.

Symmetric: xRy means yRx ? $x \geq y$ means $y \geq x$? No, if the numbers are different, it is only true one way.

Antisymmetric? Yes, Only way $x \geq y$ and $y \geq x$ is if they are equal.

Transitive: xRy and yRz then is xRz ? Yes. $x \geq y\ and\ y \geq z\ means\ x \geq z$.

So this is NOT an equivalence relation (because not symmetric) but it is a Partial Order.
(A partial order is one that is reflexive, antisymmetric, and transitive).

Does "not symmetric" mean "antisymmetric"?
Not always. Example is in 3.3 problem 29: $xRy\ iff\ x = y$^2.
Clearly this relation is not reflexive, symmetric, or transitive.
You can see it is not antisymmetric, either.
$xRy\ and\ yRx$ are not both true even if $x = y$.

On the other hand, if the relation R is given by xRy iff x=y then it is symmetric AND antisymmetric. In this case each element has an equivalence class consisting of just one element, namely itself.

Example 2 (3.3, 31) xRy if 3 divides x-y.
This is an equivalence relation (reflexive, symmetric and transitive).
Reflexive: xRx ? Yes. 3 divides x-x = 0.
Symmetric: xRy means yRx ? Yes. If 3 divides x-y then it also divides y-x because x-y is just the negative of y-x.
It is symmetric, and not the relation x=y, so it cannot be antisymmetric.
It is transitive also.
If 3 divides x-y and y-z, then 3 also divides x-z, because x-z = x-y + y-z.

Total Order and Partial Order

A relation that is reflexive, symmetric, and antisymmetric is a **partial order**.

A total order is a relation where any two elements are "**comparable**."
See Example 3 below.

This means, given x and y, either xRy or yRx.

Some partial orders are total orders.
Example: The relation xRy iff "x is less than or equal to y" on the set of real numbers.
This is reflexive, anti-symmetric, and transitive.
Any pair x, y are comparable because either xRy or yRx.

The relation xRy iff "x divides y" on the set of integers is a partial order
But *it is NOT a total order*.

Note that the same relation becomes kind of meaningless if defined on the set of
Real numbers or rational numbers. If x divides y means y = kx then any two real numbers or any two rational numbers are related.

Example 3: R is defined on P(S), the set of subsets of S. ARB iff A is a subset of B.
Note: If A is a subset of B, that includes when A=B.

If S = { 1,2,3} then the subsets {1,2} and {1,3} are not comparable. Neither is contained in the other. But you can make a chain of relations in which the subsets can be arranged in order of containment in some other subset. Example ; { } , {1}, {1,2}, {1,2,3}.

Example 4: 3.3 Exercise 37:

Define R on the set of four-bit strings (1111, 1101, 0101, etc) by xRy if x and y share a two bit string. Is R reflexive, symmetric, antisymmetric, transitive, and/or partial order?

Example: 1101 and 0111 are related because they share 11 (actually they also share 01).

Reflexive: Yes, each string has two-bit strings in common with itself.
Symmetric: Yes. If a shares two-bit strings with b, b also shares with a.
Antisymmetric: No. Relation is not equality, so it cannot be symmetric and antisymmetric.
Transitive: No. Example: 1111, 0111, 0100.
So the relation is neither a partial order nor an equivalence relation.