

Egyptian Fraction decomposition

[Exercise 25 in 2.5]

Theorem: Every nonzero fraction $\frac{p}{q} < 1$ can be written as a sum of distinct Egyptian fractions.

Proof:

Let $\frac{p}{q} < 1$ be any fraction. If $p = 1$ we are done because it is already a sum of Egyptian fractions (itself).

Idea of proof: Start with smallest n such that $1/n \leq p/q$ and then use induction assumption to write the remainder as sum of Egyptian fractions.

(Strong) Induction Assumption: Assume that any fraction i/q' with $1 \leq i < p$, and $i/q' < 1$, q' being any positive integer, is a sum of distinct Egyptian fractions.

Let $\frac{p_1}{q_1} = \frac{np-q}{nq} = \frac{p}{q} - \frac{1}{n}$ with $p_1 = np - q$, $q_1 = nq$.

Since n is smallest such that $p/q \leq 1/n$, we must have

$$\frac{p}{q} < \frac{1}{n-1} \implies \frac{p}{q} - \frac{1}{n} < \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} \leq \frac{1}{n}$$

But $\frac{p_1}{q_1} = \frac{p}{q} - \frac{1}{n}$ so we must have $p_1/q_1 < 1/n$. Note that $n \geq 2$ (because otherwise we get $p/q > 1$) so denominator $n(n-1)$ will not be zero.

Now we are ready to apply induction assumption to p_1/q_1 . As shown above it is smaller than $1/n$ so definitely smaller than 1. Also the numerator p_1 is smaller than the denominator q_1 (hence $p_1/q_1 < 1$) because $p < q$ means $np < nq$ so certainly $p_1 = np - q < nq = q_1$. Finally, numerator is smaller than p because $\frac{p}{q} < \frac{1}{n-1}$ means $(n-1)p < q$ which means $np - q < p$ but $np - q = p_1$ so $p_1 < p$.

So by induction assumption,

$$\frac{p_1}{q_1} = \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k}, \quad n_1 < n_2 < \dots < n_k.$$

We can assume they are in ascending order because they are distinct and since numerators are all equal to 1 the denominators are different and it is possible to arrange them.

But $\frac{p}{q} = \frac{p_1}{q_1} + \frac{1}{n}$ so we get

$$\frac{p}{q} = \frac{1}{n} + \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k}, \quad n_1 < n_2 < \dots < n_k.$$

The only thing left to show is that $n < n_1$. But we have

$$\frac{p_1}{q_1} < \frac{1}{n} \implies \frac{1}{n_1} + \frac{1}{n_2} + \dots + \frac{1}{n_k} < \frac{1}{n} \implies \frac{1}{n_1} < \frac{1}{n} \implies n < n_1.$$