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Some examples on strong induction to help with HW5

For the definition and explanation of strong induction, read Raji's book (Section 1.2.3, page 12) and the Discrete Math book (Section 2.5, Page 108). Raji's book has a simpler explanation.

Basically you use strong induction when you need to use not just the previous (n-th) step but other steps before that, in order to prove the n+1-th step.

1. The Lucas numbers are defined by $L_1 = 1, L_2 = 3, L_3 = 4, L_4 = 7, ...,$

   $L_n = L_{n-1} + L_{n-2} + ...$ Show that $L_n < (7/4)^n$.

   Solution: Check that it is true for $n = 1, n = 2$. We need two previous steps to prove the n-th step. Since by strong induction we assume the statement is true for all $K < n$, it could certainly be assumed true for $n-1$ and $n-2$. So we will assume that $L_{n-1} < (7/4)^{n-1}$ and $L_{n-2} < (7/4)^{n-2}$ and try to prove it for n.

   $$L_n = L_{n-1} + L_{n-2} < \left(\frac{7}{4}\right)^{n-1} + \left(\frac{7}{4}\right)^{n-2} = \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4}+1\right)$$

   Now $(7/4) + 1 = 11/4$ and $(11/4) < (7/4)^2 = 49/16$ because $11/4 = 44/16$ (multiply above and below by 4). Therefore in the last inequality we can replace $11/4$ by $(7/4)^2$. We get

   $$L_n = L_{n-1} + L_{n-2} < \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4}+1\right) < \left(\frac{7}{4}\right)^{n-2}\left(\frac{7}{4}\right)^{2} = \left(\frac{7}{4}\right)^{n}$$

   Thus we have proved the statement for n and the proof is complete.

2. (Decimal expansion of natural numbers) Prove that every positive integer n can be represented uniquely as a sum of distinct powers of 10, i.e., in the form $n = a_0 10^0 + a_1 10^1 + ... + a_k 10^k$ with integers $0 \leq a_i \leq 9$ for $i = 0, 1, 2, ..., k$.

   Solution: We take this for granted but this is something that needs to be proven!

   Example: $202 = 200 + 2 = 2(10^2) + 2(10^0)$. This is unique, i.e, the only way you can write 202 as a sum of *distinct powers* of 10.

   How do we get this? We first take the largest power of 10 that divides 22. Here it is 10^2. The remainder after division is 2 and the highest power of 10 that divides it is just 1 $(= 10^0)$.

   Now why is this unique? Suppose we have some other power of 10 in the representation. In this case the only possibility is 10^1. But then you would have to have something like 10×10 which is just 10^2. Actual proof below:

Here is the proof: Suppose we divide n by 10 and get $n = q10 + R$ where R is the remainder and q is the quotient. Note that $1 \leq R \leq 9$ by definition of the division process (Division algorithm) as satisfied by the set of integers.

If $R = 0$ then $n = 10q$ and then $n/10 < n$ and by induction hypothesis it has a unique representation in decimals.

If $R > 0$ clearly the remainder $n - R < n$. So by induction hypothesis it has a unique representation in decimals.

So either $n/10$ or $n - R$ has a unique representation as sums of distinct powers of 10.

Suppose now n is a multiple of 10 and it has another representation as sums of distinct powers of 10. Then dividing by 10 we get that $n/10$ has another representation and this is a contradiction.

If n is not a multiple of 10 then we get a unique remainder R as before and then if n has another representation $n - R$ would also have another representation. Again we get a contradiction.

This is just one way to do this problem.