10/31/2018 Fall 2018,Proofs and Problem Solving I Sitaraman

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HW5 solutions

For the definition and explanation of strong induction, read Raji's book (Section 1.2.3, page 12) and the Discrete Math book (Section 2.5, Page 108).

Raji's book has a simpler explanation.

Basically you use strong induction when you need to use not just the previous (n-th) step but other steps before that, in order to prove the n+1-th step.

HW5 DUE WEDNESDAY 10/25 BY 3pm

Problems from the books by Raji (Number Theory) and Johnsonbaugh (Discrete Mathematics).

1. Show that postage of 24 cents or more can be achieved by using only 5-cent and 7-cent stamps.

   The idea for proving it using strong induction is that in order to get a number n as a combination of 5 and 7, all we need is $n-5$ as a combination of 5 and 7. Then we can add 5 and get that n is also such a combination.

   But you cannot do this if $n-5$ is smaller than 24 because not all numbers smaller than 24 are not combinations of 5 and 7.

   So for instance using $24 = (2\times5)+(2\times7)$ we can get $29 = 24+5 = [(2\times5)+(2\times7)]+5 = (3\times5)+(2\times7)$ but $28-5 = 23$ and 23 is not a combination of 5 and 7.

   $n-5 < 24 \implies n < 29$.

   So the base cases are 24,25,26,27,28.

   These are the ones where you cannot get the required combination by adding 5 to $n-5$.

   So first we have to prove it in the base cases.

   $24 = (2\times5)+(2\times7)$; $25 = 5\times5$; $26 = (3\times7)+5$; $27 = (4\times5)+7$; $28 = 4\times7$.

   Now we are ready to prove for all n.

   Assume every $k < n$ is a combination of 5 and 7. (Actually this is more than we need! We only need $k = n-5$).

   Then $n-5 = 5x+7y$ and $n = (n-5)+5 = 5x+7y+5 = 5(x+1)+7y$.

   So assuming for all numbers smaller than n we have proved that it works for n also.

   This means we can prove it for 29, 30, 31, and so on for all ensuing natural numbers because all we need to do is to look at the number 5 less than given number and add 5 to it.

2. The Egyptians of antiquity expressed a fraction as a sum of fractions whose numerators were 1. For example, 5/6 might be expressed as

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{3}. \tag{1}$$

We say that a fraction p/q, where p and q are positive integers, is in Egyptian form if

$$\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k}$$

where $n_1, n_2, ..., n_k$ are positive integers satisfying $n_1 < n_2 < ..... < n_k$.

(a) Show that the representation (1) is not unique. That is, find another way to write 5/6 using fractions with numerator 1 and distinct denominators.

Solution:

$$\frac{5}{6} = \frac{1}{2} + \frac{1}{4} + \frac{1}{12}.$$

We have really just split 1/3 into 1/4 plus 1/12 in (1).

This is actually a general phenomenon.

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} = \frac{n+1}{n(n+1)}. \qquad (2).$$

(b) (NOT FOR TURNING IN, BUT COULD SHOW UP IN TEST) Prove by yourself or read the proof in Johnsonbaugh's book or any of the dozens of websites and papers on Egyptian fractions that every fraction p/q such that $0 < p/q < 1$ can be expressed in Egyptian fraction form.

(c) Prove (using strong or complete induction *as well as* standard or weak induction) that 1 can be written in Egyptian form using k fractions with numerator 1 for *any* natural number $k \geq 3$. For example,

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = ...$$

Proof using regular induction:

$$\text{Note that } \frac{1}{2} = \frac{1}{3} + \frac{1}{6}. \tag{3}$$

We use this repeatedly to prove by induction.

We have already done the case where 1 is a sum of 3 fractions in Egyptian form.

We also discussed why it cannot be a sum of 2 fractions.

Now we prove using regular induction for $k \geq 3$.

Looking at the pattern above, we make the following assumption (induction hypothesis) for k.

$$1 = \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n_k}$$

where the k-th fraction $1/n_k$ has an even denominator, say $n_k = 2m_k$.
Also the above expression is in Egyptian form, meaning the denominators are in ascending order and all different.
Then we split $1/n_k$ by using the decomposition of $1/2$ given by equation (3).

$$\frac{1}{n_k} = \frac{1}{2m_k} = \frac{1}{m_k}\left(\frac{1}{2}\right) = \frac{1}{m_k}\left(\frac{1}{3} + \frac{1}{6}\right) = \frac{1}{3m_k} + \frac{1}{6m_k}.$$

Now these two new fractions are smaller than $1/n_k$ which itself is smaller than all the rest, so we get a decomposition of 1 into $k+1$ fractions in Egyptian form:

$$1 = \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{3m_k} + \frac{1}{6m_k}.$$

Thus the induction process is complete.

Note: You can also do this by splitting $1/n_k$ using equation (2). The rest of the argument is exactly the same.

Proof using strong or complete induction
Just as in postage stamp problem, we can use the splitting of 1 into three fractions to go from $k-2$ Egyptian fractions to k fractions. Here is the trick:

$$1 = \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n_{k-2}} = \frac{1}{2} + \frac{1}{3} + ... + \left(\frac{1}{n_{k-2}}\right) \times 1$$

$$= \frac{1}{2} + \frac{1}{3} + ... + \left(\frac{1}{n_{k-2}}\right) \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) = \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2n_{k-2}} + \frac{1}{3n_{k-2}} + \frac{1}{6n_{k-2}}.$$

Just as before this is also an Egyptian fraction decomposition.

(d) Using the previous two results show that any nonzero rational number $r \leq 1$ can be written as a sum of unit fractions in Egyptian form in an infinite number of ways.
Solution: The idea is use what we just did for 1.
Say the fraction $r = \frac{1}{n_1} + \frac{1}{n_2} + ... + \frac{1}{n_k}$.
Multiply the last fraction by 1 and proceed as before. If you need m fractions then use the decomposition of 1 into $m-k+1$ fractions (since we can break down 1 into as many Egyptian fractions as we want).

Then the new decomposition has the first $k-1$ fractions plus the new $m-k+1$ fractions for a total of m fractions.

Note: You can do this also using equation (2) and splitting smallest fraction into two.

3. The following concern the famous Fibonacci numbers given by $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, ..., F_n = F_{n-1} + F_{n-2}....$ They are ubiquitous in nature and are related to the golden ratio, among other things.

   (a) Show that for any natural number m, F_{3m-2} and F_{3m-1} are odd while F_{3m} is even. For example, F_1 and F_2 are odd while F_3 is even.

   Solution: The first few Fibonacci numbers are 1,1,2,3,5,8,13,...

   You can see the pattern: 1 is odd, 1 is odd, 2 is even, 3 is odd, 5 is odd, 8 is even...

   For (strong) induction hypothesis assume all F_n with $n < 3m-2$ follow this pattern. This would include F_{3m-3}, F_{3m-4} and F_{3m-5}.

   Then using the relation $F_n = F_{n-1} + F_{n-2}$ and the fact that by induction hypothesis F_{3m-3}, F_{3m-4} and F_{3m-5} follow the odd, odd, even pattern you can show that F_{3m-2}, F_{3m-1} and F_{3m} also follow the same pattern.

   (b) Show that for any natural number $n \geq 4$, we have

$$\left(\frac{8}{5}\right)^{n-2} < F_n < \left(\frac{9}{5}\right)^{n-2}.$$

   [FYI: By the way, the golden ratio ϕ is 1.618... and it is between 8/5 and 9/5.
   It can be proved that $F_n/F_{n-1} \to \phi = 1.618...$ as $n \to \infty$].

   Solution:

   Check this is true for $n = 4$ and 5 because, as we will see below, we will need two steps to prove the next step.

   We need two previous steps to prove the n-th step. Since by strong induction we assume the statement is true for all $k < n$, it could certainly be assumed true for $n-1$ and $n-2$.

   So we will assume that $(8/5)^{n-1} < F_{n-1} < (9/5)^{n-1}$
   and $(8/5)^{n-2} < F_{n-2} < (9/5)^{n-2}$ and try to prove it for n.

$$\left(\frac{8}{5}\right)^{n-1} + \left(\frac{8}{5}\right)^{n-2} = \left(\frac{8}{5}\right)^{n-2}\left(\frac{8}{5}+1\right) < F_n = F_{n-1} + F_{n-2}$$

$$< \left(\frac{9}{5}\right)^{n-1} + \left(\frac{9}{5}\right)^{n-2} = \left(\frac{9}{5}\right)^{n-2}\left(\frac{9}{5}+1\right)$$

Now $(8/5) + 1 = 13/5$ and $(13/5) > (8/5)^2 = 64/25$ because $13/5 = 65/25$ (multiply above and below by 5). Therefore in the last inequality we can replace $13/5$ by $(8/5)^2$.

Similarly $(9/5) + 1 = 14/5$ and $(14/5) < (9/5)^2 = 81/25$ because $14/5 = 70/25$ (multiply above and below by 5). Therefore in the last inequality we can replace $14/5$ by $(9/5)^2$.

We get

$$\left(\frac{8}{5}\right)^n = \left(\frac{8}{5}\right)^{n-2}\left(\frac{8}{5}\right)^2 < \left(\frac{8}{5}\right)^{n-2}\left(\frac{8}{5}+1\right)$$

$$< F_n = F_{n-1} + F_{n-2} < \left(\frac{9}{5}\right)^{n-2}\left(\frac{9}{5}+1\right) < \left(\frac{9}{5}\right)^{n-2}\left(\frac{9}{5}\right)^2 = \left(\frac{9}{5}\right)^n$$

Thus we have proved the statement for n and the proof is complete.

4. (Binary expansion of natural numbers) Prove that every positive integer n can be represented uniquely as a sum of distinct powers of 2, i.e., in the form $n = 2^{i_0}+2^{i_1}+...2^{i_k}$ with integers $0 \leq i_0 < i_1 < i_2 < .... < i_k$.

   Example: $20 = 16 + 4 = 2^4 + 2^2$. This is unique, i.e, the only way you can write 20 as a sum of *distinct powers* of 2. If you use 2 or 8 you will have to repeat one of them.

   Solution: Using division algorithm, there is just one way to divide n by 2.

   You get $n = 2m + 1$ of it is odd and $n = 2m$ if it is even.

   In both cases m is uniquely defined.

   Also $m < n$ in both cases.

   So by induction hypothesis we get that m is a sum of powers of 2 in a unique way.

   Then say $m = 2^{i_0} + 2^{i_1} + ...2^{i_k}$ with integers $0 \leq i_0 < i_1 < i_2 < .... < i_k$.

   Then either $n = 2m + 1 = 2(2^{i_0} + 2^{i_1} + ...2^{i_k}) + 1 = 2^{i_k+1} + ... + 2^{i_1+1} + 2^{i_0+1} + 2^0$

   or $n = 2m = 2(2^{i_0} + 2^{i_1} + ...2^{i_k}) = 2^{i_k+1} + ... + 2^{i_1+1} + 2^{i_0+1}$. with integers $0 \leq i_0 < i_1 < i_2 < .... < i_k$.