Howard University Math Department

Pigeonhole principle, Principle of Induction

1. (Pigeonhole principle) If 100 balls are placed in 9 bags then some box contains 12 or more balls.

   Solution: The pigeonhole principle in general deals with distribution of n things in m "boxes." We can make some basic observations about how the n things are distributed by looking at the relative size of n and m. Usually n is bigger than m.

   Proof of given statement by contradiction: Suppose statement is false, and no box contains 12 or more balls. Then all boxes contain less than 12 balls. Then totally there will be at the most 11 times 9 or 99 balls. The contradiction means statement is true.

2. Prove by induction that $n^2 + n$ is always even.

   Solution: It is clearly true for 1.

   Suppose $f(n) = n^2 + n$ is even.

   We need to show that $f(n+1) = (n+1)^2 + (n+1)$ is even.

   $(n+1)^2 + (n+1) = (n^2 + 2n + 1) + n + 1 = (n^2 + n) + 2n + 2 = n^2 + n + 2(n+1)$.

   But the last term is the sum of two even numbers because by assumption $n^2 + n$ is even and clearly $2(n+1)$ is even. So it is even.

3. Prove that the sum of first n natural numbers is $f(n) = n(n+1)/2$.

   Solution: Clearly it is true for $n = 1$.

   Suppose it is true for n. i.e, the sum of first n natural numbers is $f(n) = n(n+1)/2$.

   Then we need to show that $f(n+1) = (n+1)((n+1)+1)/2 = (n+1)(n+2)/2$ is the sum of the first $n+1$ natural numbers.

   But the sum of the first $n+1$ natural numbers is

   $$(1 + 2 + 3 + ... + (n-1) + n) + n + 1 = f(n) + n + 1 = n\left(\frac{n+1}{2}\right) + (n+1)$$

   $$= (n+1)(\frac{n}{2} + 1) = (n+1)(n+2)/2 = f(n+1).$$

4. Find a pattern (and a formula) for the sum of the first n odd numbers. Can you explain the formula geometrically?

5. Prove by induction that sum of a geometric series is

$$\sum_{k=1}^{k=n} ax^{k-1} = a\left(\frac{x^n - 1}{x - 1}\right)$$

Solution: Clearly true for $n = 1$.

Suppose true for n. i.e, suppose we have

$$\sum_{k=1}^{k=n} ax^{k-1} = a\left(\frac{x^n - 1}{x - 1}\right)$$

We need to show this for $n + 1$. i.e,

$$\sum_{k=1}^{k=n+1} ax^{k-1} = a\left(\frac{x^{n+1} - 1}{x - 1}\right)$$

As in problem 3 this is the sum of the first n terms plus the $n + 1$-th term.

So it is $a\left(\frac{x^n - 1}{x - 1}\right) + ax^n = a\left(\frac{(x^n - 1) + (x - 1)x^n}{x - 1}\right) = a\left(\frac{x^{n+1} - 1}{x - 1}\right)$

6. Prove by induction that the number of subsets of a set X with N elements is 2^n. The set of all subsets is called the power sets, and is denoted as $P(X)$. So this can be written as $|P(x)| = 2^n$. Note that the empty set and the set X itself are subsets of X.

Solution:

If the set has only one element, i.e, $n = 1$, then there are two subsets: the empty set and itself. So $|P(X)| = 2^1 = 2$.

(Actually we need to start with empty set. In this case there is only one subset, and $|P(X)| = 2^0 = 1$).

Suppose it is true for n.

If X has $n + 1$ elements, fix the $n + 1-$th element, say x and divide $P(X)$ into two kinds of subsets: *All the subsets not containing* x are basically the subsets of a set of n elements, namely the first n. So there are 2^n such subsets, based on the assumption for n. *All subsets containing* x can be obtained by adding x to one of the subsets in the previous collection, and each one in the previous collection gives rise to exactly one subset in this new collection. In other words, they are in one to one correspndence and hence there are 2^n subsets in the new collection also. But every subset either contains x or doesn't. So total number of subsets is obtained by adding number of subsets in each collection. So $P(X) = 2^n + 2^n = 2(2^n) = 2^{n+1}$.

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HOMEWORK 4 (due on mon, 9/24; no late hw).

7. Give a proof by contradiction that for any subset S of 26 cards from a 52 card deck (a 52 card deck is composed of 4 suits of 13 cards each), there is a suit such that S has at least 7 cards of that suit. [This is an application of the pigeonhole principle].

   Solution:

   Suppose not. Then S has less than 7 cards in each suit, so at the most totally it would only have 4 times 6 or 24 cards. Since we assumed there are 26 cards, we get a contradiction and thus there must be 7 cards in at least one suit.

8. Prove by induction the formula for sum of the first n odd numbers that we found in problem 4.

   Solution: For $n = 1$ the sum is 1 and n^2 also equals 1.

   Assume the sum of first n odd numbers is n^2.

   So $1 + 3 + 5 + ... + 2n - 1 = n^2$.

   Then we want to show that $1 + 3 + 5 + ... + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$.

   But the first n odd numbers (ending in $2n - 1$) add up to n^2 by assumption.

   So we get $1+3+5+...+(2n-1)+(2(n+1)-1) = [1+2+3+...+(2n-1)]+2n+1 = n^2 + 2n + 1 = (n + 1)^2$ as was required.

9. Prove by induction that the sum of the first n terms of an arithmetic sequence starting with a and adding d each time is $n(2a + (n - 1)d)/2$.

   Solution: Let $S(n) = n(2a + (n - 1)d)/2$.

   Clearly it is true for $n = 1 :\ S(1) = a = 1(2a + 0d)/2 = a$.

   Suppose it is true for n. i.e, the sum of first n terms is $S(n) = n(2a + (n - 1)d)/2$.

   Then we need to show that $S(n+1) = (n+1)(2a+(n+1-1)d)/2 = (n+1)(2a+nd)/2$ is the sum of the first $n + 1$ terms.

   But the sum of the first $n + 1$ terms is

$$a+(a+d)+....+(a+(n-1)d)+(a+(n+1-1)d) = [a+(a+d)+....+(a+(n-1)d)]+(a+(n+1-1)d)$$

$$= S(n) + (a + nd) = n\left(\frac{2a + (n-1)d}{2}\right) + (a + nd) = n\left(a + \left(\frac{(n-1)d}{2}\right)\right) + (a + nd)$$

$$= a(n + 1) + n(\frac{n-1}{2} + 1)(d) = a(n + 1) + n\left(\frac{n+1}{2}\right) d = (n + 1)(2a + nd)/2.$$

10. Show, by giving a proof by contradiction, that if 40 coins are distributed among nine bags so that each bag contains at least one coin, at least two bags contain the same number of coins. [This is another application of the pigeonhole principle].

    Solution:

    If no two bags contain same number of coins, then at a minimum there will be $1+2+...+9 = 9(9+1)/2 = 45$ coins, contrary to assumption. Note that we used the formula for sum of first n natural numbers here.

11. Prove by induction that $m^3 - m$ is always divisible by 3, for every integer m. Then prove this directly.

    Solution:

    Induction: For $m = 1$ it is true because $1^3 - 1 = 0$ is divisible by any number.

    Assuming $m^3 - m$ is a multiple of 3, say $3k$, need to show that $(m+1)^3 - (m+1)$ also is a multiple of 3.

    But using binomial formula for cube of a sum we get

    $$(m+1)^3-(m+1) = m^3+3m^2+3m+1-m-1 = (m^3-m)+3m^2+3m = 3k+3m^2+3m = 3(k+m^2+m).$$

    Direct proof: $m^3 - m = m(m^2 - 1) = m(m - 1)(m + 1) = (m - 1)m(m + 1)$.

    This is a product of 3 consecutive numbers, at least one of whom must be a multiple of 3.