Howard University Math Department

Homework 1 Brief Solutions

1. Prove Pythagoras' theorem geometrically using the two pictures shown on 8/20.

   Solution:

   The main idea: Since the two squares are identical of area $(a+b)^2$, if we remove equal areas from both the remainder would also be equal.

   First, using SAS (side-angle-side) theorem note that all the triangles in the two squares are congruent (meaning they are exactly the same) because two sides a and b and the included angle (90 degrees) are all the same. This can be proved using Euclid's first four postulates.

   Second, using the fact that in a triangle the sum of angles is 180 (this is equivalent to the Parallel postulate also known as Euclid's fifth postulate – see notes titled "Notes about Euclid's fifth postulate" posted 8/24) and the sum of angles on a line is 180 we can show that all the angles inside the quadrilateral in the middle of the picture on LHS are right angles. To be precise, if you add the angles in any of the triangles you get 180. So the two acute angles in any of those triangles add up to 90. Now take any of the angles in the quadrilateral. The two angles on both sides of it outside the quadrilateral are equal to the two acute angles in the triangles. So their sum is 90 and since the angle in question is also on the same line and sum of angles on a line is 180 it must equal 90 degrees also. [The fact that the sum of the angles on a line is 180 is really just a matter of definition. The total angle at any point by definition is 360, and when you draw a line through a point you get two equal angles on both sides, so each must be 180].

   Finally each side of that quadrilateral is of length c because they are all on the hypotenuses of triangles with sides a and b.

   So now we have that the remaining area (colored gray) in picture 1 is a square of area c^2. On the other hand the remaining (gray) area in picture 2 is equal to $a^2 + b^2$. Since two areas must be equal, as mentioned at the top, we get $c^2 = a^2 + b^2$.

   Incidentally note that we have proved Pythagoras' theorem assuming only Euclid's five postulates. In other words, assuming the first four, if fifth postulate is true, so is Pythagoras' theorem. So Pythagoras' theorem is also equivalent to Euclid's fifth postulate/axiom.

PROBLEM 2 NEXT PAGE

2. Using only picture 1 (LHS) above give a proof of Pythagoras' theorem using geometry and algebra.

   Solution: Geometrically, we have shown that the area you obtain inside the square of area $(a+b)^2$ after removing four right triangles of sides a, b is c^2. Thus

$$\begin{aligned} &\text{Area of square minus the area of the 4 triangles} \\ &= (a+b)^2 - 4\left(\frac{ab}{2}\right) \\ &= (a^2 + b^2 + 2ab) - 2ab = a^2 + b^2 \\ &= \text{Area of inner square} = c^2. \end{aligned}$$