Proofs and Problem Solving

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Class Notes

More definitions

1. Negative of a statement: Opposite.

   If one of them is true the opposite is false.

1. Example: Which of the following are opposite of “All men are created equal” ?
   a. All women are created equal.
   b. All men are not created equal.
   c. Some men are better than others.
   d. Not every man can run like Usain Bolt.

Answer: only (b) is negative. (c) would be a negative if it said it said “Some men are created better than others.”

In symbols, negative of A -> B is A -> “NOT B”

More definitions (2)
Converse of a statement

Example: Where there is a will, there is a way.

Converse: If there is a way, there must have been will.

Converse may not always be true.

In symbols, converse of A -> B is B -> A.

What is the opposite of this statement?

More definitions (3) Contrapositive

- A -> B is equivalent to NOT B -> NOT A.
- Why do we need it?

Sometimes contrapositive is easier to prove. See Page titled “Proof by contrapositive”

- Example: Contrapositive of “Where there is a will there is a way”:
- If there is no way there must have been no will.

Example from calculus

- If a function is differentiable then it is continuous.
- What are the negative, converse and the contrapositive of this statement?
- Is the converse true? If not, how do we prove that?
- Negative: If a function is differentiable then it is NOT continuous
- Converse: If it is continuous then it is differentiable
- Contrapositive: If it is not continuous then it is not differentiable
- The given statement is true means its negative is false.
- The converse is false (see next page).
- The contrapositive of any true statement is also true.

Proof by counter-example

- To prove that something is *not true,* it is enough to show that it is not true even in *one case.*
- So for instance it is enough to show that there is one function that is continuous but not differentiable, in order to show that the converse of “If f is differentiable then it is continuous” is false.
- One example is the absolute value function

  f(x) = |x| that has a pointed edge where f’(x) is undefined.

Proof by contrapositive

- Prove: If the sum of angles in a triangle is 180, then at least one angle is greater than or equal to 60 degrees.
- This is easier to prove by showing that the contrapositive is false.
- To prove A -> B you assume NOT B and show NOT A is true.
- Here B = "at least one angle is > or = to 60"
- NOT B = "each angle is < 60"
- But if you assume each angle is smaller than 60, then you get that their sum is less than 180, which is not true.
- This type of proof is called "Proof by contrapositive."

Examples (1)

- Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- Everyone who was ever voted MVP of the NBA finals has won that final.
- Counter-example: Jerry West who was the first one to win MVP of finals was in losing team.
- When women lead nations they don't go to war.
- Counter-example: Cleopatra, Queen Elizabeth, Margaret Thatcher, Indira Gandhi

Examples (2)

- Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- If you want to walk the shortest distance between two points you must walk in a straight line.
- Contrapositive: If you don't walk in a straight line you will end up walking farther between the same two points. – easy to see!
- If the sky is dark then the sun is not shining.
- Contrapositive: If sun is shining then sky is bright – obvious!

Proof by contradiction

- By *slightly* changing the argument in the proof by contrapositive we can get a proof by contradiction.
- Instead of proving "NOT B -> NOT A" you assume B is false, A is true, and get a contradiction (that A is false). Sometimes the contradiction maybe that something other than A is false.
- For example, you assume that sum of angles is 180 AND each angle is less than 60. Then you get a contradiction in the form of sum being less than 180.
- In general we assume the opposite of *what we want to prove* and get a contradiction to one of our assumptions or some well-established fact.

Example:(Proof by Contradiction) Euclid's proof on prime numbers

Statement: (3rd cy BC) ***There are infinitely many prime numbers***

Proof: (using the method ***Proof by Contradiction***)

1. Suppose there are only finitely many, say p_1, p_2, p_3,…., p_n are ALL the n natural numbers that are divisible by 1 and themselves. Let P be their product.
2. Then P+1 leaves a remainder of 1 when you divide by any of the primes, so it is divisible exactly only by 1 and itself!
3. Therefore P+1 is a prime number that is not equal to any of the other prime numbers.
4. Therefore there are more than n prime numbers.
5. But we assumed that there are only n of them!

The contradiction means the statement should be true.