9/18/2025 Fall 2025, Proofs and Problem Solving I Test 1 v1 Solutions Sitaraman

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1. (15 points) State the converse, contrapositive and negative of the following statement about functions taking real numbers to real numbers that are differentiable in a closed interval $[a, b]$:

   If the function $f(x)$ has a local maximum or minimum in $[a, b]$ then the derivative is zero for some $x \in [a, b]$.

   Converse: If the derivative of $f(x)$ is zero for some $x \in [a, b]$ then the function f has a local maximum or minimum in $[a, b]$

   Contrapositive: If the derivative of $f(x)$ is not zero for any $x \in [a, b]$ then the function f has no local maximum or minimum in $[a, b]$.

   Negative: If the function $f(x)$ has a local maximum or minimum in $[a, b]$ then the derivative is not zero for any $x \in [a, b]$.

   NOTE: **Negative of** $A \implies B$ **is** $A \implies \neg B$ **or** $A \wedge \neg B$.

   Alternately: The function $f(x)$ has a local maximum or minimum in $[a, b]$ and the derivative is not zero for any $x \in [a, b]$.

2. (16 points) Use truth table to check that $p \implies q$ and $(\neg p) \vee q$ are equivalent.

   Solution:

p	q	$p \implies q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

3. (16 points) State the negative of the following statements:

   a) Either the climate is not changing or most scientists are lying.

   b) $\forall x \in [0, \infty), x^2 \leq 2^x$.

   Solution:

   a) The climate is changing and most scientists are not lying.

   b) $\exists x \in [0, \infty), x^2 > 2^x$.

4. (16 points) Prove using the contrapositive: If x is a rational number and y is irrational, then $x + y$ is irrational.

   Solution:

   Proof: Assume $x + y$ is rational. We need to show that either x is irrational or y is rational. If x is irrational, we are done.

   Suppose x is rational, then since difference of two rational numbers is also rational, $(x + y) - x = y$ is rational.

   NOTE: $x + y$ **can be rational without either** x **or** y **being rational!**

   Example: $\sqrt{2} + (2 - \sqrt{2}) = 2$.

5. (16 points) Prove or give counterexample: Domain is set of positive real numbers.

$$\forall x \in (0, \infty),\ 2^x - 1 \leq x^2. \quad (Note:\ x > 0).$$

   Solution:

   This is false. It is true for $x = 1, 2, 3, 4$ but not true for $x \geq 5$. You can also see it is not true for numbers between 0 and 1 such as 0.5 and in fact using logarithms you can show it is not true for all those numbers.

6. Given $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5\}$, $C = \{1, 5, 6\}$ and the universal set is $U = \{1, 2, 3, 4, 5, 6\}$ check if following are true *for* A, B , C.

   NOTE: To disprove something, enough to give ONE counterexample. So if a statement is not true for some sets, it is false. But to establish it to be true, need to prove for ALL sets. So you can only disprove something using the specific sets A, B, and C.

   a) (8 points) $A \cup (B \cap C) = (A \cup B) \cap C$.

   (b) (6 points) $\overline{A \cup C} = \overline{A} \cap \overline{C}$

   (c) (8 points) $A - (B \cap C) = (A - B) \cup C$

   Solution:

   a) $A \cup (B \cap C) = \{1, 2, 3, 4, 5\} \neq (A \cup B) \cap C = \{1, 5\}$.

   (b) is true by DeMorgan's law.

   (c) $A - (B \cap C) = A \neq (A - B) \cup C = \{1, 2, 5, 6\}$