

**Howard University Math Department**

1. (15 points) State the converse, contrapositive and negative of the following statement about functions taking real numbers to real numbers that are differentiable in a closed interval  $[a, b]$ :  
(Need not check if statement is true)

If the function  $f(x)$  has a local maximum or minimum in  $[a, b]$  then the derivative is not zero for some  $x \in [a, b]$ .

Converse: If the derivative of  $f(x)$  is not zero for some  $x \in [a, b]$  then the function  $f$  has a local maximum or minimum in  $[a, b]$ .

Contrapositive: If the derivative of  $f(x)$  is zero for all  $x \in [a, b]$  then the function  $f$  has no local maximum or minimum in  $[a, b]$ .

Negative: If the function  $f(x)$  has a local maximum or minimum in  $[a, b]$  then the derivative is zero for all  $x \in [a, b]$ .

NOTE: **Negative of  $A \implies B$  is  $A \implies \neg B$  or  $A \wedge \neg B$ .**

Alternately: The function  $f(x)$  has a local maximum or minimum in  $[a, b]$  and the derivative is zero for all  $x \in [a, b]$ .

2. (16 points) Use truth table to check if  $p \implies q$  and  $(\neg p) \wedge q$  are equivalent.

Solution:

| $p$ | $q$ | $p \implies q$ | $\neg p \wedge q$ |
|-----|-----|----------------|-------------------|
| $T$ | $T$ | $T$            | $F$               |
| $T$ | $F$ | $F$            | $F$               |
| $F$ | $T$ | $T$            | $T$               |
| $F$ | $F$ | $T$            | $F$               |

They are NOT equivalent.

3. (16 points) State the negative of the following statements:

a) The climate is changing and most scientists are not lying.

b)  $\exists x \in [0, \infty), x^2 > 2^x$ .

Solution:

a) Either the climate is not changing or most scientists are lying.

b)  $\forall x \in [0, \infty), x^2 \leq 2^x$ .

4. (16 points) Prove using the contrapositive: If  $x$  is a rational number and  $y$  is irrational, then  $xy$  is irrational. Assume they are all nonzero.

Solution:

Proof: Assume  $xy$  is rational. We need to show that either  $x$  is irrational or  $y$  is rational. If  $x$  is irrational, we are done.

Suppose  $x$  is rational, then since quotient of two rational numbers is also rational,  $(xy)/x = y$  is rational.

NOTE:  $xy$  can be rational without either  $x$  or  $y$  being rational!

Example:  $\sqrt{2} \times \sqrt{2} = 2$ .

5. (16 points) Prove or give counterexample: Domain is set of positive real numbers.

$$\forall x \in (0, \infty), 2^x \leq (x+1)^2. \quad (\text{Note : } x > 0).$$

Solution:

This is false. It is true for  $x = 0, 1, 2, 3, 4, 5$  but not true for  $x \geq 6$ . You can also see it is true for numbers between 0 and 1 such as 0.5 and in fact using calculus you can show it is not true for all those numbers.

6. Given  $A = \{1, 2, 3, 4\}$ ,  $B = \{3, 4, 5\}$ ,  $C = \{1, 5, 6\}$  and the universal set is  $U = \{1, 2, 3, 4, 5, 6\}$  check if following are true for  $A$ ,  $B$ ,  $C$ .

NOTE: To disprove something, enough to give ONE counterexample. So if a statement is not true for some sets, it is false. But to establish it to be true, need to prove for ALL sets. So you can only disprove something using the specific sets  $A$ ,  $B$ , and  $C$ .

a) (8 points)  $A \cap (B \cap C) = (A \cap B) \cap C$ .

(b) (6 points)  $\overline{A \cup B} = \overline{A} \cap \overline{B}$

(c) (8 points)  $A - (B - C) = (A - B) - C$

Solution:

a)  $A \cap (B \cap C) = \{ \} = (A \cap B) \cap C$ .

(b) is true by DeMorgan's law.

(c)  $A - (B - C) = \{1, 2\} \neq (A - B) - C = \{2\}$