Proofs and Problem Solving

Logical Relations and Proofs

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9/3/2025
Class Notes

Example : Graph theory

- In any group of 6, there will be 3 strangers or 3 acquaintances.
- Negative: In any group of 6, there will neither be 3 strangers nor 3 acquaintances.
- Converse: If there are always 3 strangers or 3 acquaintances in a certain number of people, then there must be 6 people.
- Contrapositive: If there are neither 3 strangers nor 3 acquaintances, then there must not be 6 people.

Proof by counter-example

- To prove that something is *not true,* it is enough to show that it is not true even in *one case.*

- So for instance it is enough to show that there is one function that is continuous but not differentiable, in order to show that the converse of "If f is differentiable then it is continuous" is false.
- One example is the absolute value function

$f(x) = |x|$ that has a pointed edge where $f'(x)$ is undefined.

Exercises Chapter 1, Section 3.
Proof by Counterexample

1. $\forall n \in \mathbb{N}, n^2 + n + 41$ is a prime number

(For any natural number, $n^2 + n + 41$ is prime)

2. $\nexists x \in \mathbb{Q}$ such that $x^2 + (x-1)^2 = x^2 + 1.$

(No rational number satisfies this equation)

Proof by contrapositive

- Prove: If the sum of angles in a triangle is 180, then at least one angle is greater than or equal to 60 degrees.
- This is easier to prove by showing that the contrapositive is false.
- To prove A -> B you assume NOT B and show NOT A is true.
- Here B = “at least one angle is > or = to 60”
- NOT B = “each angle is < 60”
- But if you assume each angle is smaller than 60, then you get that their sum is less than 180, which is not true.
- This type of proof is called “Proof by contrapositive.”

Proof by contrapositive
Example 1 (chapter 1, section 3)
and Optional Exercise

If n^2 is even, then n is even

Contrapositive: If n is odd, then n^2 is odd.

Prove this by letting $n = 2k + 1$ and then squaring both sides. Show that RHS will always be odd.

Optional : Show that 3, 5, 7 are the only prime triples (3 consecutive numbers that are primes).

Proof by contrapositive
Example 2

Prove by contrapositive: If the product of three positive numbers is greater than 27, then at least one of them is greater than 3.

Also write the negative and converse of this statement.

Contrapositive: Not B -> Not A

Not B: None of them is greater than 3 (or ALL are < or = 3)

Then product is < or = 27.(or not > 27) -- this is NOT A.

Thus we have proved A -> B by using contrapositive.

NOTE: Need to PROVE that if a,b,c are all < or = 3 then abc < or = 27 !

Negative and converse on next page.

Example 2-contd

Prove by contrapositive: If the product of three positive numbers is greater than 27, then at least one of them is greater than 3.

Also write the negative and converse of this statement.

Negative: A -> NOT B

If the product of three positive numbers is greater than 27, then none of them is greater than 3.

Converse: B -> A

If at least one of three positive numbers is greater than 3 then the product is greater than 27.

NOTE: Converse is not always true! Can you find an example?

Proof by contra: Examples (1)

- Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- Everyone who was ever voted MVP of the NBA finals has won that final.
- When women lead nations they don’t go to war.

Proof by contra: answers

- Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- Everyone who was ever voted MVP of the NBA finals has won that final.
- Counter-example: Jerry West who was the first one to win MVP of finals was in losing team.
- When women lead nations they don't go to war.
- Counter-example: Cleopatra, Queen Elizabeth,
  Margaret Thatcher, Indira Gandhi

Examples (2)

Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- If you want to walk the shortest distance between two points you must walk in a straight line.
- If the sky is dark then the sun is not shining.

Examples (2) answers

- Prove the following statements (or their negative, if the statement is not true) by contrapositive or counter-example:

- If you want to walk the shortest distance between two points you must walk in a straight line.
- Contrapositive: If you don’t walk in a straight line you will end up walking farther between the same two points. – easy to see!
- If the sky is dark then the sun is not shining.
- Contrapositive: If sun is shining then sky is bright – obvious!

Proof by contradiction

- By *slightly* changing the argument in the proof by contrapositive we can get a proof by contradiction.

- Instead of proving “NOT B -> NOT A” you assume B is false, A is true, and get a contradiction (hence proving that A must be false). Sometimes the contradiction maybe that something other than A is false. Basically you will get some absurd result by assuming B is false and A is true.

- In general we assume the opposite of *what we want to prove* and **get a contradiction to one of our assumptions or some well-established fact.**

Proof by contradiction
-- but not contrapositive

If n+1 is even then n is odd.

Suppose n is even and n+1 is even.

Then their difference is even

So 1 is even – absurd / not true.

The contradiction shows that both cannot be even.

Proof by contrapositive –example 1

- If x is a rational number and y is irrational, then x+y is irrational.

Proof: Assume x+y is rational.

- Since x is rational,
- And difference of two rational numbers is also rational,
- (x+y) – x = y is rational – contradicting hypothesis!

More exercises

- Prove by contrapositive: If there is congested traffic with a lot of cars during rush hour it must mean not everyone is taking public transit trains to work.

Answer: Note: NOT B -> NOT A but not directly!

Proof by contrapositive is NOT the same as writing the contrapositive of a statement!

If all are riding trains then they are not driving, so there is not a lot of cars during rush hour causing congestion.

Another exercise on Proof by Contradiction

Show that if $n > 1$ then $(n+1)^2 > 4n$ for all n.
First look at 1,2,3, etc., to see for yourself.
Your conclusion must be something other th
an the opposite of $n > 1$.

Answer: Start assuming NOT B

i.e, that $(n+1)^2$ < or = 4n

for at least one value of $n > 1$.

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Another exercise on Proof by Contradiction -contd

Answer: Start assuming NOT B

i.e, that $(n+1)^2$ < or = $4n$

for at least one value of $n > 1$.

Simplifying we get

n^2+2n+1 < or = $4n$ for at least one $n > 1$.

So $n^2-2n+1 = (n-1)^2$ < or = 0 for at least one $n > 1$
but the square of a positive number

is always > 0 -- contradiction

Example:(Proof by Contradiction) Euclid's proof on prime numbers

Statement: (3rd cy BC) ***There are infinitely many prime numbers***

Proof: (using the method ***Proof by Contradiction***)

1. Suppose there are only finitely many, say p_1, p_2, p_3,…., p_n are ALL the n natural numbers that are divisible by 1 and themselves. Let P be their product.
2. Then P+1 leaves a remainder of 1 when you divide by any of the primes, so it is divisible exactly only by 1 and itself!
3. Therefore P+1 is a prime number that is not equal to any of the other prime numbers.
4. Therefore there are more than n prime numbers.
5. But we assumed that there are only n of them!

The contradiction means the statement should be true.

Two challenge problems (for those who want it)

1. Show that there are infinitely many primes of the form 4x-1.

(i.e, Show that the sequence 3,7,11,15,…has infinitely many primes in it).

You can use the same method as for all natural numbers but will need a slight modification.

2. Show that you can find two prime numbers differing by any amount, for example, you can find two prime numbers that are a billion numbers apart.

Some practice problems-1

1. Write the negative, contrapositive and converse of the following (where applicable):
   a) Not all cakes are created equally tasty.
   b) If roses are red and violets are blue then sunflowers are yellow.
   c) If a function is discontinuous then it is undefined at least at one point.

   Is statement 1c true? Is its converse true?

Some practice problems - 2

Prove the following by contradiction or contrapositive. If the statement is not true then prove that it is false using counterexample.

1. (A+B)UC = A+(BUC) for all sets A,B, and C where + means intersection.
2. For all real numbers x and y,

   if $x + y \geq 2$ then either $x \geq 1$ or $y \geq 1$.

3. If 100 balls are placed in 9 boxes, then some box contains at least 12 balls.

Some practice problems - 3

Chapter 1, 3.9 (a) : Check argument:

$r \implies \neg s, t \implies s \; ; \; r \implies \neg t$

Idea: Use contrapositive of $r \implies s$

Chapter 1, 4.11 : $\nexists \; a, b, c \in \mathbb{Z}$ consecutive, even such that $a^2 + b^2 = c^2 -$ True or False? Prove.

False: Solving $(2k)^2 + (2k+2)^2 = (2k+4)^2$ we get $k = -1,3$ from which we get $(-2,0,2) \; and \; (6,8,10)$ as counterexamples