Proof and Problem Solving
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Class Notes

Why do we need this course? (added 6-17-19)

- Why do we need this, other than for requirements?
- Basketball/music analogy:
- Finished with drills, now for the skills
- Some may need more time – we all have different ways of thinking
- Not to show how to solve a problem, but to help you learn to solve ANY problem

Solving Problems

- IDEA → SOLUTION → PROOF OF SOLUTION

- WHAT YOU LEARN:
- PRESENTATION
- ANALYSIS
- DEMONSTRATION

EXAMPLE 1

- WHAT IS THE FLAW IN THE ARGUMENT ?

- REAGAN CUT TAXES
- ECONOMY GREW
- THEREFORE TAX CUTS GROW ECONOMY

- Flaw: Need to consider other factors in economic growth / show how tax cuts result in growth.
- Correlation is not causation!

EXAMPLE 2

- PEOPLE WHO LIVE NEAR OIL REFINERIES GET CANCER MORE THAN PEOPLE WHO DON'T. THEREFORE OIL REFINERIES CAUSE CANCER
- SIMILAR : VACCINES CAUSE AUTISM
- Similar flaws as in Example 1. Here the first statement is true, the second is not.

NUMBER THEORY EXAMPLE

- 1+ 2 = 3 = (2 X 3)/2
- 1+2+3 = 6 = (3 X 4)/2
- 1+2+3+4 = 10 = (4 X 5)/2

- THEREFORE 1+2+3+….+N = N(N+1)/2

- This is true but need to prove it.
- Will prove this by INDUCTION later.
- Basically, we will show that you can always go from N to N+1. i.e, if it is true for 1 then it is true for 2 and if it is true for 2 then it is true for 3 and so on for all natural numbers.

PROOF BY COUNTEREXAMPLE

- FERMAT NUMBERS
- $F_n = 2^{2^n} + 1$
- IS PRIME FOR n = 1,2,3,4 BUT NOT FOR 5 BECAUSE 641 DIVIDES F_5

What is a *Proof ?*

- A series of logical statements together called an ***argument***

that results in establishing the truth of a mathematical statement of fact called **lemma** or **theorem**.

- The statements should each be true and one statement should lead to the next statement.

- They should use only the **definitions** and **axioms** that have been previously stated and the **hypotheses** that have been made in the lemma or theorem to be proved.

Example: Pythagoras Theorem

- In any right triangle, $c^2 = a^2 + b^2$

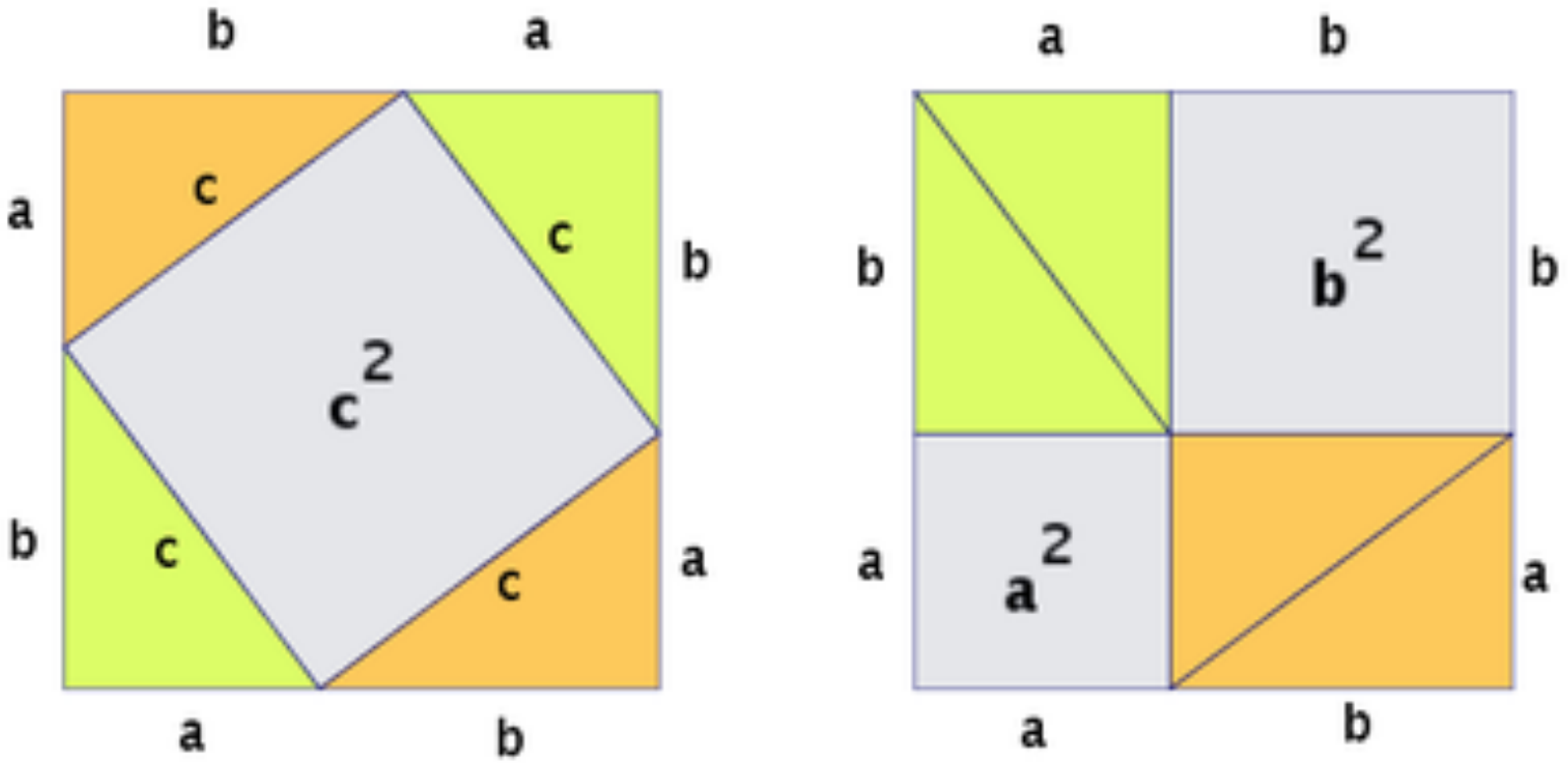
Here a, b are the sides, c is hypotenuse.

- How do you prove this?

- Is it enough to check some right triangles?

Geometrical proof

Picture thanks to https://math.stackexchange.com



Questions

1. Write down the step by step proof using the two pictures, stating the assumptions you made and axioms that you used.

2. Algebraic Proof: From the picture on the LHS (left hand side) give another proof using both geometry and algebra as well.

PARTY PROBLEM

GIVEN A PARTY OF SIX PEOPLE

EITHER THERE ARE THREE PEOPLE
WHO KNOW EACH OTHER
OR
THERE ARE THREE WHO DON’T KNOW

APPETIZER

HOW MANY PEOPLE ARE NEEDED

SO THAT THERE WILL BE

TWO WHO KNOW OR
TWO WHO DON’T KNOW EACH OTHER?

ANSWER

Need only two people.

Either they know each other or they don’t.

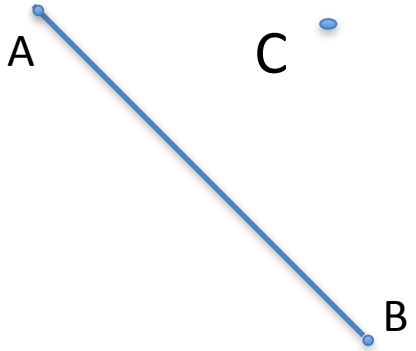
CASE OF N = 3, 4

WHAT HAPPENS WHEN THERE ARE
3 OR 4 PEOPLE IN PARTY?

ARE THERE THREE WHO KNOW EACH OTHER
OR THREE WHO DON'T KNOW EACH OTHER?

Answer for N = 3

- No. Example for 3:



This is a “GRAPH.” It represents three people and the line between A and B indicates they know each other.

Answer for N = 4.

- Again answer is NO. A square with vertices A,B,C,D representing four people is an example where you have neither 3 that know each other nor 3 that do not know each other.

- In what follows a solid line will indicate two people know each other. A dashed line - - - - will indicate two who do not.

HOMEWORK

DO THE SAME AS PREVIOUS SLIDE

FOR N = 5

Answer: The pentagon is a counterexample.

It has no triangle, meaning no three know each other. It also has no set of three who don't know each other because every person (point) knows two others (adjacent points).

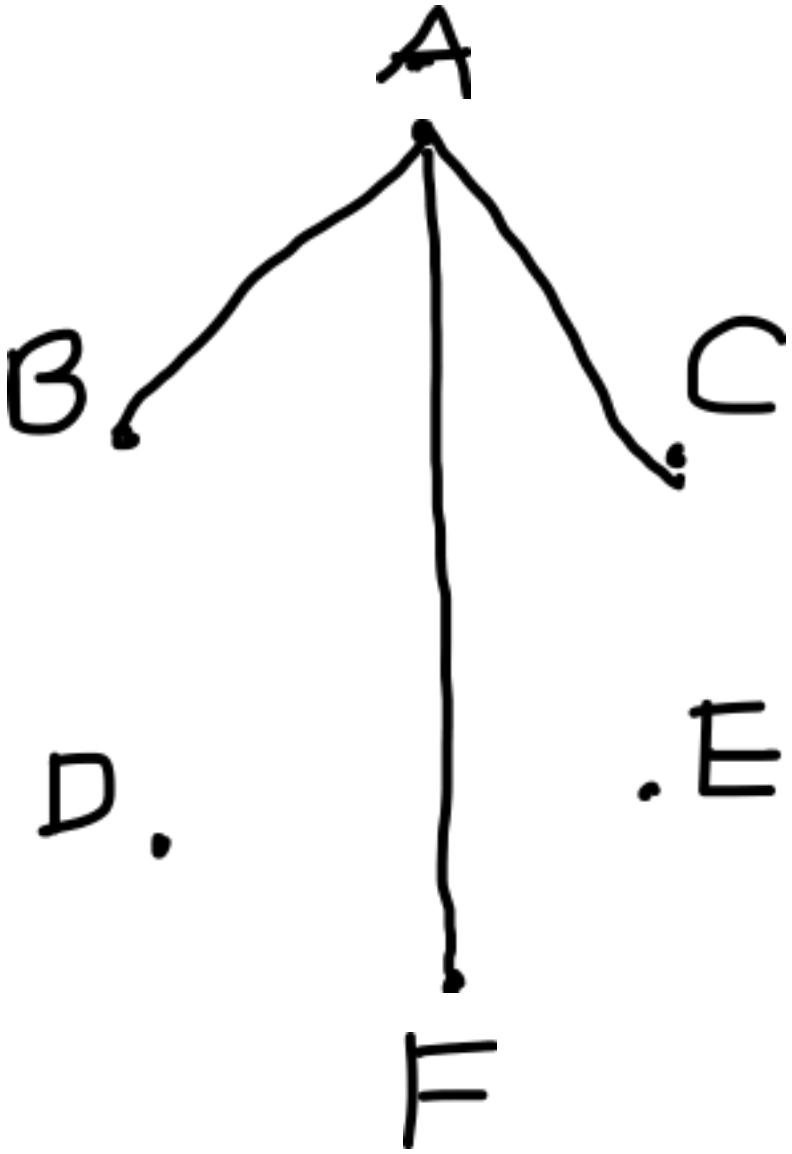
SOLUTION OF PARTY PROBLEM

Start with A.

Among B,C,D,E,F, it must know three or not know three.

Doesn't matter which three it knows or which three it doesn't, we will get three lines (either solid or dashed).

Among B,C,F, if any two know each other we get a solid triangle. If no two know each other we get a dashed triangle.



Practice problem 1

What is the flaw in following arguments?

1. Scientists say the earth has warmed – average temperatures have gone up. How come this winter was so cold?

2. If everyone drives everywhere we won't need trains.

Answers on next page.

Practice problem 1

1. Scientists say the earth has warmed – average temperatures have gone up. How come this winter was so cold?

Answer: Average doesn't mean daily temperatures will also vary uniformly

2. If everyone drives everywhere we won't need trains.

Answer: Trains are not just for passengers, also freight

A number theory argument

$2^2-1 = 3$, a prime number.

$2^3-1 = 7$, a prime.

$2^5-1 = 31$, a prime

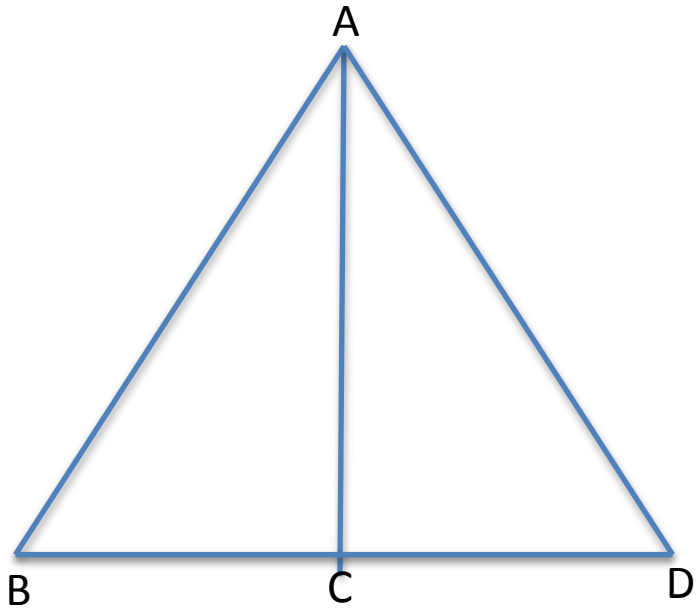
$2^7-1 = 127$, a prime

Therefore 2^p-1 is always a prime whenever p is a prime number.

This is false. Counterexample: $2^{11} – 1 = 2047$

is not prime. It is 23 times 89.

Proving a geometrical theorem-1
(Thanks to mathsisfun.com)

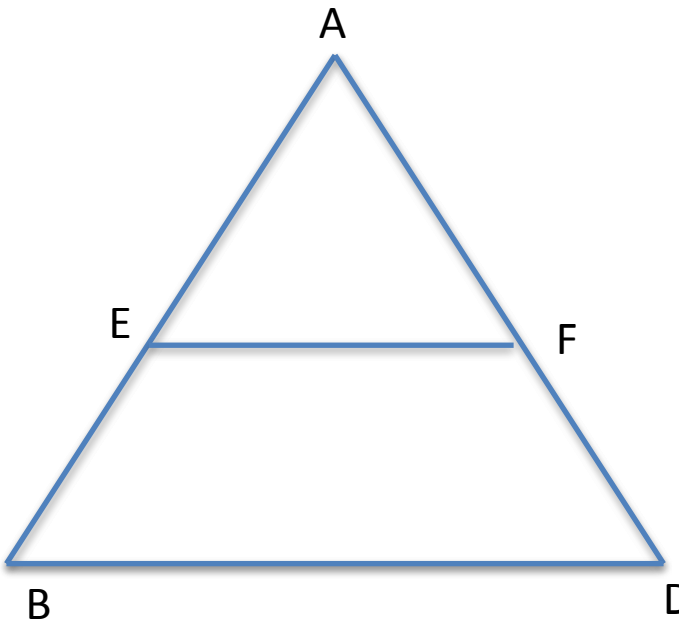


Isosceles triangle with AC a perpendicular bisector of BD.
i.e, BC = CD

Show that AB/BC = AD/CD

ABC and ACD are actually congruent, which means they are not just similar but all the corresponding sides and angles are equal. So not only are the ratios equal, AB= Ad, BC = CD.

Proving a geometrical theorem-2 (Thanks to mathsisfun.com)

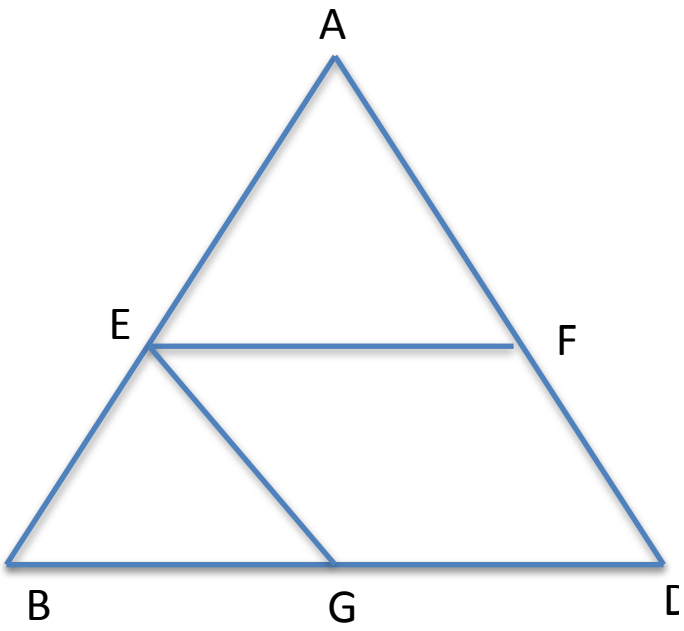


ABD is any triangle.

EF is parallel to BD.

Show that AE/EB = AF/FD

Proof of geometrical theorem-2

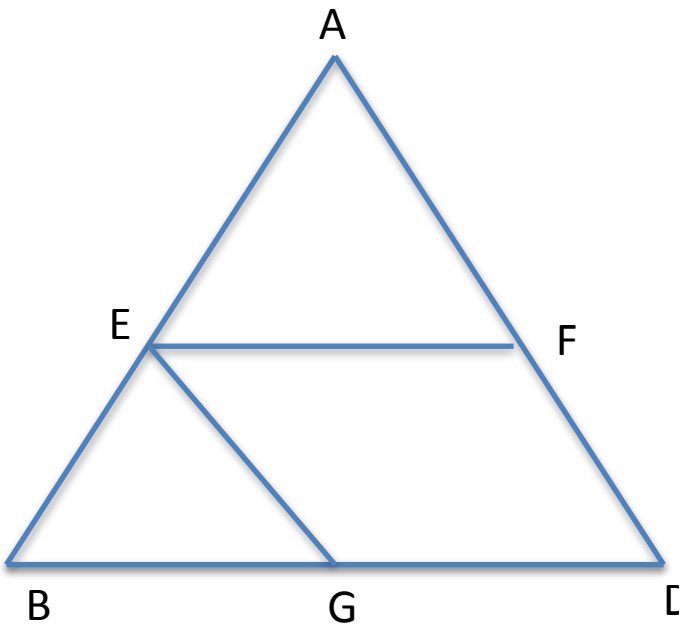


Proof: Draw EG parallel to FD. Then each angle of AEF is equal to an angle of EBG. Also note that EG = FD.

Angle AFE equals Angle EGB Because EF is parallel to BD AND EG is parallel to FD.

The sides opposite them are AE and EB.

Proof of geometrical theorem-2 continued



Triangles AEF and EBG are similar.

Corresponding sides means sides opposite to angles that are equal.

Proof: Similarly Angle AEF equals Angle EBG because EF is parallel to BD.

The sides opposite them are AF and EG. Remember EG equals FD because EF is parallel to BD and EG is parallel to FD.

So AE/EB = AF/EG = AF/FD Because ratios of corresponding sides are equal and EG = FD.

Another party problem

- What is the smallest number of people in a party such that 3 people know each other or 2 don't ? Answer is 3.

- This *Ramsey number* is denoted R(3,2).

- Can you find and prove a general formula for R(n,2) for any natural number n > 1 ?

Answer

- R(n,2) = n

Proof: Answer is n because in a group of n people, *either all of them know each other or there is at least 2 who don't know each other.*

For example R(4,2) = 4 and not 2 or 3. If there are just two people it is possible they know each other. So we have *neither 4 that know each other NOR 2 that don't know each other.* Similarly if there are 3 people it is possible they all know each other and we won't have 4 that know each other or 2 that don't. **Key is that in *all possible configurations of the set of people, 4 should know each other or 2 should not know each other.***