11-3-2025 Notes, Proofs and Problem Solving 1

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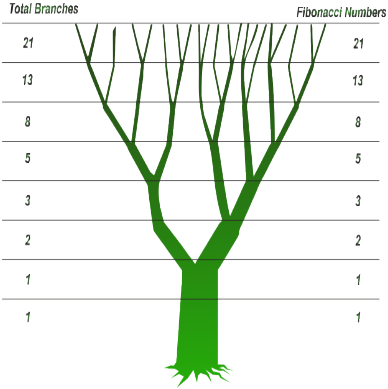
Outline

Fibonacci Sequence

1,1,2,3,5,8,13,...

Let a_n denote the $n-$th term for $n = 1, 2, 3, ...$
(You can think of a_n as a function $f : \mathbb{N} \to \mathbb{C}$).
$a_1 = 1, a_2 = 1, a_n = a_{n-1} + a_{n-2}$ *for* $n \geq 3$.
Actually $a_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n)$ where
$\alpha = (1+\sqrt{5})/2, \beta = (1-\sqrt{5})/2$.



Geometric Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence $\{4, -1, \frac{1}{4}, -\frac{1}{16}, \frac{1}{64}, ...\}$

Answer: This is a **geometric** sequence.
Each term is *multiplied* by same number,
called common ratio and denoted r.
To get to $n-$th term you need to multiply $n-1$ times.
Here $r = -1/4$ and $a_n = 4r^{n-1}$.

Geometric sequences are the **discrete** analogs of the exponential function. (and vice versa).

Example 1 – Geometric sequence

Find a formula for the *n*-th term:

$$\frac{3}{5}, \frac{2}{3}, \frac{20}{27}, \frac{200}{243}, \ldots$$

Answer for Example 1

Answer: Common ratio is $r = 10/9$. First term is 3/5.

To get to $n-$th term you need to multiply $n - 1$ times.

Here $r = -1/4$ and

$$a_n = \frac{3}{5} r^{n-1}.$$

Check by plugging in!

Arithmetic Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence

$$\{5, 8, 11, 14, 17, ...\}$$

This is an **arithmetic** sequence.
Each term is *increased* (or decreased) by same number, called common difference and denoted by d.
To get to $n-$th term you need to add $n - 1$ times.
Here $d = 3$ and $a_n = 5 + 3(n - 1) = 3n + 2$.

This is also same as formula for linear function. Linear function increases by the value of the slope each time x is increased by 1.

Example 2 – Arithmetic Sequence

$$1, 0, -1, -2, -3, \ldots$$

Write a formula for the n-th term.

The common difference is -1.

To get to $n-$th term you need to add $n-1$ times.

Here $d = -1$ and $a_n = 1 + (-1)(n-1) = 2 - n$.

Oscillating Sequence

Problem 11.1.14: Find a formula for a_n of the following sequence

$$\{1, 0, -1, 0, 1, 0, -1, 0, ...\}$$

This in example of an **oscillating** sequence.
In fact you might recognize it as the values of the sine function at

$$\frac{\pi}{2}, \frac{2\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{2}, \ldots$$

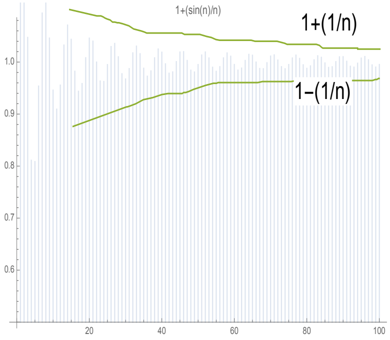
$$a_n = \sin \frac{\pi}{2} n.$$

You can also use the cosine function or do it using powers of -1.

Example for squeeze theorem

Sequences : part of a function. Graph of $b_n = 1 + (\sin(n)/n)$ at 1,2,3...,99,100 :

Each vertical line represents a value of the sequence.
The values of b_n are squeezed between $a_n = 1 + (1/n)$ and $c_n = 1 - (1/n)$.
By squeeze theorem (see below), since $a_n \to 1, b_n \to 1$, and $-1 \leq \sin(n) \leq 1 \implies a_n \leq b_n \leq c_n$, we get $b_n \to 1$.



Limit of a sequence – definition

Finding limits of sequences is similar to finding limits of functions.

$$\lim_{n\to\infty} a_n = L \textit{ if } \forall\epsilon\ \exists\ N \textit{ for which } |a_n - L| < \epsilon\ \forall\ n \geq N$$

If L exists the sequence is said to be **convergent.**

FACT: If the function $f(x) \to L$ as $x \to \infty$ then $f(n) \to L$ also, as $n \to \infty$

Proof: exercise

Limit of a sequence : Squeeze theorem

SQUEEZE THEOREM FOR SEQUENCES

If $a_n \leq b_n \leq c_n$ for $n \geq N$ and $a_n \to L, c_n \to L$, then $b_n \to L$ also.

Proof: $a_n \to L, c_n \to L$ means for any $\epsilon > 0$, $|a_n - L| < \epsilon$ for $n \geq N_1$ for some N_1 and $|c_n - L| < \epsilon$ for $n \geq N_2$ for some N_2. Choosing N to be the bigger of N_1, N_2, we have:

$$L - \epsilon < a_n < L + \epsilon, L - \epsilon < c_n < L + \epsilon, \forall n \geq N.$$

$$\implies L - \epsilon < a_n, c_n < L + \epsilon, \forall n \geq N.$$

But $a_n \leq b_n \leq c_n$ means $L - \epsilon < a_n \leq b_n \leq c_n < L + \epsilon, \forall n \geq N$. Thus $L - \epsilon < b_n < L + \epsilon, \forall n \geq N$ for any $\epsilon > 0$ and limit as $n \to \infty$ of b_n is also L.

Limit of a sequence – boundedness

CONVERGENCE OF BOUNDED, MONOTONIC SEQUENCES

BOUNDED: terms of sequence are always above or below a certain value. Either $a_n \leq M$ or $a_n > m$ for all n.
MONOTONIC: Either increasing or decreasing.

Exercise 1

Write down a formula for the $n-$th term (a_n) of the sequence and find its limit, if it exists:

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.$$

Is it bounded? If so what is the bound?
Is it monotonic?

Exercise 1

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Is it bounded? If so what is the bound?
Is it monotonic?

Exercise 1 – bounded

It is bounded because $a_n < 1$ for all n.

We know this because the denominators are always bigger than the numerators.

In fact 1 is the smallest upper bound. Anything bigger than 1 is also an upper bound. We will see in next slide that 1 is the limit.

Exercise 1 – monotonic

It is also monotonic increasing, i.e, always increasing.
Each term is smaller than the next.
$a_n = n/(n+1)$. Putting $n+1$ instead of n we get the next term
$a_{n+1} = (n+1)/(n+2)$.

$a_n < a_{n+1}$ because $n/(n+1) < (n+1)/(n+2)$.
To prove this cross multiply and check that $n(n+2) < (n+1)^2$.
Note: This can be done because all terms are positive.

Exercise 1 – using squeeze theorem

The **squeeze (sandwich) theorem** is really redundant here but you can still use it in the following way:

$$1 - \frac{1}{n} < \frac{n}{n+1} = 1 - \frac{1}{n+1} < \frac{n}{n} = 1$$

Both $1 - \dfrac{1}{n}$ on the left and 1 on the right go to 1,
so the sequence is squeezed or sandwiched between two sequences
both of which go to 1. So the limit of given sequence is also 1.

Exercise 1 – Using L'Hospital's rule

You can also use L'Hospital's because both numerator and denominator go to ∞.

$$\lim_{x\to\infty} \frac{x}{x+1} = \lim_{x\to\infty} \frac{x'}{(x+1)'} = \lim_{x\to\infty} \frac{1}{1} = 1.$$

Since the real valued function $\dfrac{x}{x+1} \to 1$ as $x \to \infty$, the sequence $\dfrac{n}{n+1} \to 1$ as well.

Limit of a sequence – boundedness

THEOREM: EVERY BOUNDED, MONOTONIC SEQUENCE IS CONVERGENT.

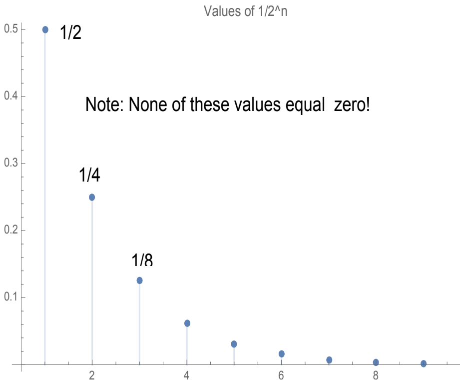
Proof is based on the COMPLETENESS AXIOM for the set of real numbers:
Every non-empty, bounded above subset of $\mathbb{R}$ *has a least upper bound*

Limit of a sequence – boundedness – proof

PROOF: Assume that sequence is monotonic increasing. (Proof is similar if decreasing).
Suppose L is a least upper bound for a_n. Then for any $\epsilon > 0$ we must have $a_N > L - \epsilon$ for some N.
Since sequence is increasing, $a_n > a_N > L - \epsilon$ for all $n \geq N$.
Also $a_n < L$ because L is upper bound, so $L - \epsilon < a_n < L$ for all $n \geq N$.
Hence $|a_n - L| < \epsilon, \forall n \geq N$ and so $a_n \to L$ as $n \to \infty$.

Infinite Sums with Finite Value

IS IT POSSIBLE TO ADD AN INFINITE SET OF NUMBERS AND GET A FINITE VALUE AS ANSWER?
ANSWER: YES!! Kind of happened with integrals: Gabriel's horn has finite volume.
YOU CAN THINK OF SUM ON RIGHT AS AN APPROXIMATION TO AN INTEGRAL!



Infinite Sums with Finite Value – 2

SO WHAT IS THE SUM?
NOTICE THE PATTERN:

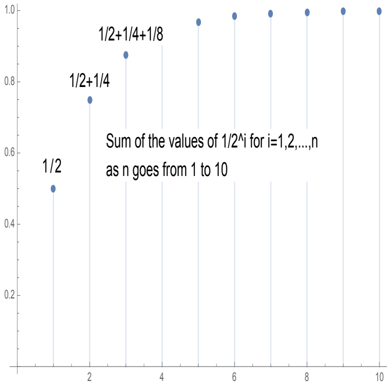
$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}=1-\frac{1}{4}$

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{3}{4}+\frac{1}{8}=\frac{7}{8}=1-\frac{1}{8}\ldots$

In general,

$\frac{1}{2}+\frac{1}{4}+\ldots.+\frac{1}{2^n}=1-\frac{1}{2^n}$

The reason this is happening is that *each time we are adding half the distance between the previous sum and 1.*



Geometric Sequence sum

FORMULA FOR SUM OF GEOMETRIC SEQUENCE

$$S(n) = 1 + r + r^2 + \dots r^n$$
$$\implies rS(n) = r + r^2 + \dots + r^{n+1}$$
$$\implies (1-r)S(n) = 1 - r^{n+1}$$
$$\implies S(n) = \frac{1}{1-r}\left(1 - r^{n+1}\right)$$

THE ABOVE FORMULA IS VALID FOR ANY $r \neq 1$!!

Example 1 – Geometric sequence

Suppose you deposit \$ 5000 in an account each month for 10 years at a return of $i = 0.08$ (8 percent)
The amount from the first month results in $5000(1+i)^{10}$ after 10 months.
The amount from the second month results in $5000(1+i)^{9}$ after 9 months.
and so on .....
The last amount gives no return, so results in just 5000.
What is the total amount that you would have after 10 months?

Answer for Example 1

Use formula for geometric series :

$1 + r + r^2 + ... + r^{n-1} = \dfrac{1 - r^n}{1 - r}$.

Here we have $r = 1 + i = 1.08$.

$$S(10) = 5000 + 5000r + 5000r^2 + ... + 5000r^{10}$$

$$\implies S(10) = 5000(1 + r + ... + r^{10})$$

$$\implies S(10) = 5000 \times \frac{1 - r^{11}}{1 - r}$$

$$\implies S(10) = 5000 \times \frac{1 - (1.08)^{11}}{1 - 0.08}$$

$$\implies S(10) = 5000 \times \frac{1.3316}{0.08} = 83,227.44\$$$

What is a series?

A SERIES IS A SEQUENCE OF SUMS

We call the limit of the series as the limit of the sum.

Note that the limit of a_n is not the same as limit of $\sum a_n$!

Infinite sum or geometric series sum

Using the formula

$$1 + r + r^2 + \ldots + r^{n-1} = \frac{1 - r^{n+1}}{1 - r}$$

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = \frac{1}{1-(1/2)}\left(1 - \frac{1}{2^{n+1}}\right)$$

$$= 2\left(1 - \frac{1}{2^{n+1}}\right) = 2 - \frac{1}{2^n}$$

$$\implies \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n} \to 1 \; as \; n \to \infty.$$

We say that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1.$

Question on series

Find an ϵ such that the geometric series sum s_n is within 0.0001 of 2.
Here s_n is the sum of the sequence 1, 1/2, 1/4, ...$1/2^n$...

Infinite geometric series – properties

Just as with integrals, $\displaystyle\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty}\sum_{k=1}^{n} a_k$.

We say the limit of $\displaystyle\sum_{k=1}^{n} a_k$ approaches L if $\displaystyle\left|\sum_{k=1}^{n} a_k - L\right| \to 0$ as $n \to \infty$. Or in more fancy words, for some N, the distance $\displaystyle\left|\sum_{k=1}^{n} a_k - L\right| < \epsilon$ for any $\epsilon > 0$ whenever $n > N$.

Here we have $\displaystyle\left|\sum_{k=1}^{n} a_k - L\right| = \left|\left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}\right) - 1\right| = \left|\left(1-\frac{1}{2^n}\right)-1\right| = \frac{1}{2^n} \to 0$

Infinite geometric series – properties

In general, if $-1 < r < 1$ then the sum of the geometric series

$$1 + r + ... + r^n = \frac{1 - r^{n+1}}{1 - r} \to \frac{1}{1 - r}.$$

If $r > 1$ or $r < -1$ (i.e, $|r| > 1$) then the series diverges.
If the geometric sequence starts with a number a instead of 1, the sum equals

$$a + ar + ... + ar^n = a(1 + r + r^2 + ... + r^n) = a\left(\frac{1 - r^{n+1}}{1 - r}\right) \to \frac{a}{1 - r}.$$

A telescoping series

The sum of the sequence $\dfrac{1}{k(k+1)} = \dfrac{1}{2}, \dfrac{1}{6}, \dfrac{1}{12}, \ldots$ is an example of a telescoping sum.

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$+ \left(\frac{1}{3} - \frac{1}{4} \right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n} \right) + \left(\frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{n+1}$$

So this sum is also finite, approaching 1.

Note that this is close to the sum of $1/k^2$.

Example 2: telescoping series

ANOTHER TELESCOPING SUM (11.2.48)

$$\sum_{k=2}^{\infty} \frac{1}{n^3 - n}$$

Note that the sum starts with a_2, not a_1.
[In fact, you can ignore first several terms when testing convergence and finding sum].
Start by factoring $n^3 - n$ to get $n(n^2 - 1) = n(n-1)(n+1)$.
Next use partial fraction decomposition.

Example 2: telescoping series – page 2

$$\frac{1}{n^3-n} = \frac{-1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}.$$

You can try to write out this sum and cancel terms but it is easier if you modify it first.

Write

$$\frac{-1}{n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)} = \frac{-1}{2n} + \frac{-1}{2n} + \frac{1}{2(n-1)} + \frac{1}{2(n+1)}$$

and then rearrange it as $\frac{1}{2}\left(\frac{1}{(n-1)} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n}\right)$.

$$\text{So } \sum_{k=2}^{\infty} \frac{1}{n^3-n} = \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{(n-1)} - \frac{1}{n}\right) + \frac{1}{2}\sum_{k=2}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n}\right)$$

Example 2: telescoping series – page 3

The two sums on the right are telescoping sums.

In fact you can show that the first is $\frac{1}{2}\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right)$

and that the second is $-\frac{1}{2}\left(\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right)\right)$

in exactly the same way as we evaluated the telescoping sum $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

Example 2: telescoping series – page 4

Finally

$$\sum_{k=2}^{\infty} \frac{1}{n^3 - n}$$

$$= \frac{1}{2}\left(\lim_{n\to\infty} 1 - \frac{1}{n}\right) - \frac{1}{2}\left(\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right)\right)$$

$$= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

If sum is finite, terms go to zero

You might have noticed that in the sum of geometric series, the terms get very small, in fact they are almost equal to zero, eventually.
Without this it would not be possible to add infinitely many and still get a finite sum.
This is stated as a theorem (proven fact) below:

Theorem: If $\sum_{k=1}^{\infty} a_k = L$, a finite quantity, then the $n-$th term $a_n \to 0$.

Proof: If sum is finite, terms go to zero

The following proof of this theorem is slightly different from book proof.
Here only to show you how you logically establish what I just mentioned above.

Proof: As we saw in previous page, $\sum_{k=1}^{\infty} a_k = L$ means that

$$\left|\sum_{k=1}^{\infty} a_k - L\right| < \epsilon \text{ for any } \epsilon \text{ however small for all } n \text{ eventually.}$$

If $|(a_1 + a_2 + ... + a_{n-1} + a_n) - L| < \epsilon$ for all $n > N$,
then in particular it is true for both n and $n + 1$.

Proof: If sum is finite, terms go to zero – continued

So $|(a_1 + a_2 + ... + a_{n-1} + a_n) - L| < \epsilon$ and
$|(a_1 + a_2 + ... + a_{n-1} + a_n + a_{n+1}) - L| < \epsilon$.
Remember the triangle inequality $|A + B| \leq |A| + |B|$.
(If you haven't seen it just assume it true for the moment).
Let $S_n = a_1 + a_2 + ... + a_{n-1} + a_n$,
$A = S_{n+1} - L = a_1 + a_2 + ... + a_{n-1} + a_n + a_{n+1} - L$, $A + B = a_{n+1}$.
Then $B = L - (a_1 + a_2 + ... + a_{n-1} + a_n) = L - S_n$.

Proof: If sum is finite, terms go to zero – conclusion

Note that $|L - S_n| = |-(S_n - L)| = |S_n - L|$ because both positive and negative of same number have same absolute value.
By above conclusion we have $|S_n - L| < \epsilon$ AND $|S_{n+1} - L| < \epsilon$.
Now $|A + B| \leq |A| + |B| \implies |a_{n+1}| \leq |S_{n+1} - L| + |L - S_n| = |S_{n+1} - L| + |S_n - L| < \epsilon + \epsilon = 2\epsilon$.
So a_{n+1} can also get as small as we want.

Divergence Test

NOTE: The reverse doesn't work, though.
The terms can go to zero, but still the sum can be infinite!
As shown in book, $1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + ..$
goes to infinity.
You can kind of see it using the integral of $1/x$.
But if a_n does not go to zero, then *the series is guaranteed to diverge.*
This is called the **divergence test.**

Divergence Test – example

WARNING: THE SUM OF THE SERIES AND THE LIMIT OF THE TERMS OF THE SEQUENCES ARE TWO DIFFERENT THINGS.

The sequence $\{n/(n+1)\} \to 1$ i.e, $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \to 1$ but the sum $1 + \frac{1}{2} + \frac{2}{3} + \ldots + \frac{n}{n+1} + ..$ does not go to 1.
In fact by the divergence test the sum is not finite because the $n-$th term $a_n = n/(n+1) \to 1$ and not to 0.
We can actually prove it goes to ∞ because the terms all are about 1 eventually and so you are just adding 1 repeatedly.

EXAMPLE 3: DIVERGENCE TEST

$$\sum_{n=1}^{\infty} e^{1/n}$$

This diverges by the divergence test.

Since $1/n \to 0$ as $n \to \infty$, we get that $e^{1/n} \to e^0 = 1$ using the continuity of e^x.

Thus a_n does not go to zero and the series diverges.

Oscillating Series

A SEQUENCE THAT HAS DIVERGENT SUM BUT NOT INFINITE SUM

Consider $1, 0, -1, 0, 1, 0, -1, 0, 1, 0, ...$
This sequence oscillates and hence never goes to 0.
(In fact it is $a_n = \sin(n\pi/2)$ as seen in class).
You can see that the sum goes like $1, 1, 0, 0, 1, 1, 0, 0, ...$
This does not have a limit because the distance of this sequence of sums from any fixed number never goes to zero.
In other words, it doesn't stay close to any one particular number, no matter how many terms of $1, 0, -1, 0, ...$ you add up.
The sums are always alternating between 1 and 0 instead of staying close to one of the two.

Example 4: Geometric Series

ANOTHER GEOMETRIC SEQUENCE

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n}$ converge?
If so, what is the sum?

Example 4: Geometric Series – answer

$n-$th term $\dfrac{(-1)^n}{3^n} \to 0.$

This can be seen by using the squeeze or sandwich theorem:
$\dfrac{-1}{3^n} \leq \dfrac{(-1)^n}{3^n} \leq \dfrac{1}{3^n}.$

The sequences on the left and right both go to 0 so the given sequence will also go to 0.

You can also say that this is a geometric sequence with $r = -1/3$.

Since $-1 < r < 1$ the sequence will go to zero.

Since the $n-$th term goes to zero, the series *may* converge.

Example 4: Geometric Series – continued

In this case, we can show not only that it does converge, but also find the sum because it is a geometric series.

The sum goes as $\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \ldots$ with the first term being $a = \frac{-1}{3}$ and the common ratio being the same, i.e, $r = \frac{-1}{3}$.

Since $|r| = 1/3 < 1$, the series converges and the sum is given by

$$\frac{a}{1-r} = \frac{-1/3}{1-(-1/3)} = \frac{-1/3}{4/3} = \frac{-1}{3} \times \frac{3}{4} = \frac{-1}{4}.$$