Howard University Math Department

1. Bijection from $\mathbb{N}$ (natural numbers) to $2\mathbb{N}$ (even numbers).

   Solution:

   NOTE: FOR ALL OF THESE PROBLEMS, MANY SOLUTIONS ARE POSSIBLE.

   Define $f(n) = 2n$.

   1-1: $f(n_1) = f(n_2) \implies 2n_1 = 2n_2 \implies n_1 = n_2$.

   Onto: Given any even number $2m \in 2\mathbb{N}$ we have $2m = f(m)$ so it is onto.

2. Bijection from $\mathbb{N}$ (natural numbers) to $\mathbb{Z}$ (integers).

   Solution:

   NOTE: FOR ALL OF THESE PROBLEMS, MANY SOLUTIONS ARE POSSIBLE

   Define separately for odd and even numbers: $f(2n) = n, f(2n+1) = -n$. Here n runs through all natural numbers plus 0 so that we get all the even and odd natural numbers in the domain. We include $n = 0$ to give $1 = 2(0) + 1$.

   NOTE: This map is same as saying $f(x) = x/2$ if x is even, $f(x) = -(x-1)/2$ is x is odd.

   1-1: Easier to check separately for odd and even (image of an even number is always positive, so it cannot equal image of an odd number): $f(2n_1) = f(2n_2) \implies n_1 = n_2 \implies 2n_1 = 2n_2$; $f(2n_1+1) = f(2n_2+1) \implies -n_1 = -n_2 \implies n_1 = n_2 \implies 2n_1 + 1 = 2n_2 + 1$.

   Onto: Given any positive integer n we have $f(2n) = n$ and for any negative integer $-n$ we have $f(2n+1) = -n$ so it is onto.

3. Prove that there is no bijection between $I_n = \{1, 2, 3, ..., n-1, n\}$ and I_m if $n \neq m$. Also prove that there is no bijection between $I_n = \{1, 2, 3, ..., n-1, n\}$ and $\mathbb{N}$.

   Solution:

   Proof: Assume opposite, namely there is a bijection. If codomain is I_m with $m > n$ or the codomain is $\mathbb{N}$ then there are more elements than there can be images. Remember that a bijection is a function and a function can only map each element to a unique element in codomain. So there can be atmost n images and in both cases there will be elements in the codomain that have no pre-images. So the map cannot be onto. This contradiction proves there is no bijection in these cases.

   If codomain is I_m with $m < n$ then there are more elements than there can be images. In this case, more than one element in domain have to be mapped to same element in the codomain because otherwise some elements will not have any image. Again, because it is a function each element must have an image. So the map cannot be 1-1. This contradiction proves there is no bijection in these cases as well.