4/17/2026 Spring 2026, Proofs and Problem Solving Using Squeeze Theorem Sitaraman
Howard University Math Department

Using Squeeze Theorem

Question: Show that $1+\frac{\log n}{n} \to 1$ as $n \to \infty$ using squeeze theorem.

Solution:

Enough to show that $\frac{\log n}{n} \to 0$ as $n \to \infty$.

When $n > 1$, $\frac{\log n}{n} > 0$. So we have

$$0 < \frac{\log n}{n}.$$

To use squeeze theorem we need another function that bounds it on the other side, in other words the other function must always be bigger than given function. One good candidate is $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

Need to prove that, for suitably large n,

$$0 < \frac{\log n}{n} \leq \frac{1}{\sqrt{n}}.$$

If this is true, then by squeeze theorem $\frac{\log n}{n} \to 0$ as $n \to \infty$ because both the function on the left (the zero function) and the function on the right (namely $\frac{1}{\sqrt{n}}$) $\to 0$ as $n \to \infty$.

[We will see in next class how $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$.]

We will prove this by using an calculus trick. It involves two steps:

Step 1: To show $f(x) \geq g(x)$ in an interval, enough to show that $f(x) - g(x) \geq 0$ in that interval.

Step 2: To show that $f(x) - g(x) \geq 0$ in an interval, enough to show that it is at or above 0 at the beginning of that interval and thereafter increases. To show it increases, enough to show that derivative is positive in that interval.

Here $f(x) = \log n/n$ and $g(x) = 1/\sqrt{n}$.

Step 1: Enough to show $\frac{1}{\sqrt{n}} \geq \frac{\log n}{n} \to 0$ as $n \to \infty$. Since n is positive in $[1, \infty)$ we can multiply both sides by n without changing inequality. So enough to show that $\sqrt{n} \geq \log n$ for $n \in [1, \infty)$.

Step 2: First we figure out where $\sqrt{n} - \log n$ increases in $[1, \infty)$. For this we use derivative and see where derivative is positive.

$$(\sqrt{x} - \log x)' = \frac{1}{2\sqrt{x}} - \frac{1}{x} \geq 0 \implies x \geq 4.$$

Note that we are using the real valued function here, but since the sequence is part of it, it also satisfies the same inequality.

So derivative is positive in $[4, \infty)$ means $\sqrt{n} - \log n$ is increasing in $[4, \infty)$. At $n = 4$ we can check that $\sqrt{4} - \log 4 = 2 - 1.39 = 0.61$ approximately so it is positive. Since it is positive at 4 and increasing afterwards, we have proved that $\sqrt{n} \geq \log n$ for $n \in [4, \infty)$.