

1. (15 points) Prove by breaking down into two cases: If  $n$  is an integer, then  $n^2 - n$  is always an even integer.

Note: You can prove this easily by saying  $n^2 - n = n(n - 1)$  and that either  $n$  or  $n - 1$  is even. In either case, 2 divides the product which is  $n^2 - n$ .

But you need to prove this by cases by saying: either  $n$  is even and  $n = 2k$  and  $n^2 - n = 4k^2 - 2k = 2(2k^2 - k)$  or  $n$  is odd and  $n = 2k + 1$  and  $n^2 - n = 4k^2 + 4k + 1 - 2k - 1 = 4k^2 + 4k - 2k = 2(2k^2 + 2k - k)$ . In both cases we see that  $n^2 - n$  is a multiple of 2.

2. (15 points) Prove by contradiction:  $\sqrt{5}$  is irrational.

Solution:

Assume  $\sqrt{5}$  is rational. Let  $\sqrt{5} = m/n$  with  $m, n$  having no common factors. Then squaring both sides we get  $5 = m^2/n^2$ .

From this we get  $5n^2 = m^2$  which means  $5|m^2 \implies 5|m$  due to the prime factorization theorem. Let  $m = 5k$  then  $m^2 = 25k^2$  and plugging this back into  $5n^2 = m^2$  we get  $5n^2 = 25k^2 \implies n^2 = 5k^2$ . From this we get  $5|n^2$  and as before this gives  $5|n$ .

But then 5 is a common factor of  $m, n$  and we get the contradiction.

3. (20 points) Prove that the interval  $(-1, 1)$  is uncountable by making a bijection from  $(-1, 1)$  to the real numbers.

[Hint: use the  $\tan x$  function.]

Solution: This was proved in class. See notes from 3/24.

4. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough. You can quote a theorem as part of the proof but saying "this was proved in class" is not enough.
  - a) (5 points) The set of all rational numbers with odd denominators is a countable set.
  - b) (5 points) A proper subset of any set cannot have a bijection with the whole set.
  - c) (5 points) The set of points  $(x, y)$  on the plane where  $x, y$  are integers is countable.

Solution:

a) True. This is just a subset of the rational numbers: Any such number can be written as a fraction. Note that it is not all of the rational numbers.  $1/2$  for example is not in

this set, as it has an even denominator. We proved in class that subsets of countable set are countable.

(b) False. It is only true for finite sets, because in that case they must have the same number of elements. The set of even numbers is a proper subset of the natural numbers, yet you can make a bijection between them by mapping each natural number to twice itself.

(c) True. This is just the Cartesian product of the integers with itself, and we proved in class that integers are countable and cartesian product of countable sets is countable.

5. (15 points) Prove using induction that  $3^n \geq 2^n + n$  for all  $n \in \mathbb{N}$ .

For  $n = 1$  we have  $3^1 \geq 2^1 + 1 = 3$ .

Assume true for  $n$ .

Then  $3^n \geq 2^n + n$ .

Multiply both sides by 2. Get  $2 \times 3^n \geq 2 \times 2^n + 2n$ .

But  $3 \times 3^n > 2 \times 3^n \implies 3^{n+1} > 2 \times 3^n \geq 2^{n+1} + 2n$ .

Now  $2n \geq n + 1$  because  $n \geq 1$ .

Therefore

So  $3^{n+1} > 2 \times 3^n \geq 2^{n+1} + 2n \geq 2^{n+1} + n + 1$  which is the statement for  $n + 1$ .

6. (20 points) The Fibonacci numbers are defined by

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, \dots, F_n = F_{n-1} + F_{n-2} \dots$$

Prove using strong induction that  $F_n \leq 2^{n-1}$  for all natural numbers  $n \geq 1$ .

Solution:

The base case will be  $n = 1$  and  $2$  because Fibonacci numbers are built by adding two at a time.

$$F_1 = 1 \leq 2^{1-1} = 1.$$

$$F_2 = 1 \leq 2^{2-1} = 2.$$

Suppose  $F_k \leq 2^{k-1}$  for all  $k < n$ . Then it is true for  $k = n - 1$  and  $k = n - 2$  as well.

So  $F_{n-1} \leq 2^{n-1-1} = 2^{n-2}$  and  $F_{n-2} \leq 2^{n-2-1} = 2^{n-3}$ .

Adding,

$$F_{n-1} + F_{n-2} \leq 2^{n-2} + 2^{n-3} = 2^{n-3}(2 + 1) < 2^{n-3}(4) = 2^{n-3}2^2 = 2^{n-1}.$$

But  $F_n = F_{n-1} + F_{n-2}$ , so we have proved that  $F_n \leq 2^{n-1}$ .

7. (extra credit 20 points) Prove that  $\frac{n}{2^n} \rightarrow 0$  as  $n \rightarrow \infty$ . You must use proof using basic definition of limits or use a suitable theorem that was proved in class. Use of L'Hospital rule will not get credit.

Solution: We proved in class that  $\log n/n \rightarrow 0$  as  $n \rightarrow \infty$ . This actually follows from that because, if you let  $2^n = m$ , then  $n \log 2 = \log m$ , and  $\frac{n}{2^n} = \frac{1}{\log 2} \times \frac{\log m}{m} \rightarrow 0$  as well.

You can also prove it goes to 0 directly by showing that  $\frac{n}{2^n} \leq \frac{\sqrt{2^n}}{2^n}$ , for instance, for all  $n \geq 4$ . Then use squeeze theorem.

You can also prove it using squeeze theorem by using  $\frac{n}{2^n} \leq \frac{1}{n}$ . But for this you need to show  $n^2 \leq 2^n$  as  $n$  gets bigger. This can be done either using induction or by using calculus, similar to how we showed  $\log n \leq \sqrt{n}$ , in class.