Howard University Math Department

Instructions:

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 30 minutes

Please read the questions carefully before answering

1. (15 points) Prove by breaking down into two cases: If n is an integer, then $n^2 - n$ is always an even integer.

2. (15 points) Prove by contradiction: $\sqrt{5}$ is irrational.

3. (20 points) Prove that the interval $(-1, 1)$ is uncountable by making a bijection from $(-1, 1)$ to the real numbers.

   [Hint: use the $\tan x$ function.]

4. Check if following are true. To disprove something, enough to give ONE counterexample. But to establish it to be true, need to provide a proof. Examples are not enough. You can quote a theorem as part of the proof but saying "this was proved in class" is not enough.

   a) (5 points) The set of all rational numbers with odd denominators is a countable set.

   (b) (5 points) A proper subset of any set cannot have a bijection with the whole set.

   (c) (5 points) The set of points (x, y) on the plane where x, y are integers is countable.

5. (15 points) Prove using induction that $3^n \geq 2^n + n$ for all $n \in \mathbb{N}$.

6. (20 points) The Fibonacci numbers are defined by

   $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, ..., F_n = F_{n-1} + F_{n-2}....$

   Prove using strong induction that $F_n \leq 2^{n-1}$ for all natural numbers $n \geq 1$.

7. (extra credit 20 points) Prove that $\dfrac{n}{2^n} \to 0$ as $n \to \infty$. You must use proof using basic definition of limits or use a suitable theorem that was proved in class. Use of L'Hospital rule will not get credit.