3/18/2026 Spring 2026, Proofs and Problem Solving Test 2 Solutions Sitaraman

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1. (18 points) Check which of the following relations between the real numbers $\mathbb{R}$ and the integers $\mathbb{Z}$ are well-defined functions. If it is a well-defined function, say if it is 1-1 and if it is onto. Explain your answers.

   (a) The floor function : $f(x)$ is the greatest integer $n \leq x$.

   Example: $f(1.1) = 1, f(-3.7) = -4, f(0) = 0$, etc.,

   (b) $f(x) = \dfrac{x}{|x|}$.

   (c) (Codomain is also real numbers for this) $f(x) = y$ where y is found by solving the equation $x = y^2$

   (a) This is a function because it maps each real number to one and only one integer. It is not one-one. In fact it is a step function, with each interval $[n, n+1)$ being mapped to n where $n \in \mathbb{Z}$. It is onto because for each integer you can find real numbers above it that map to that integer.

   (b) This is not a function because it is not defined on 0. It maps positive real numbers to 1 and negative reals to -1. It is a function if the domain is $\mathbb{R} - \{0\}$.

   (c) This is not a function because, for each x you have two choices for y, namely the positive and negative square roots of x. Also y is not defined for negative values of x.

2. (16 points) Find the range of the following functions, and check if they are onto. Explain.

   (a) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$.

   (b) $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^x$.

   Solution:

   (a) Range is $\mathbb{R}$. It is onto because every real number has a cube root.

   (b) Range is $(0, \infty)$. So it is not onto. The function never touches or goes below x-axis. e being a positive number, e^x is always positive for both positive and negative real numbers. It is also not zero because $e^x = 0 \implies x = \ln 0$ which is undefined. You can also see this from the definitions of e^x using power series, limits, etc.,

3. (20 points) Prove the following statements if true. Give counterexample if false.

   a) The composition of two injective (i.e, 1-1) functions is always injective.

   b) A 1-1 function from a set to itself is always onto.

   Solution:

a) True. $f \circ g(x_1) = f \circ g(x_2) \implies f(g(x_1)) = f(g(x_2))$. Since f is 1-1, we must have $g(x_1) = g(x_2)$. But since g is 1-1, we must have $x_1 = x_2$. Therefore, $f \circ g$ is 1-1.

b) False, The map $f(x) = e^x$ is 1-1 but not onto as we saw above.

4. (15 points) Prove the following: If R is an equivalence relation on a set X and $Y \subseteq X$ is any subset of X then R is also an equivalence relation on Y. Prove also that the equivalence classes of Y are of the form $Y \cap X_i$ where X_i are the equivalence classes of X.

   Solution:

   Proof: All the 3 properties, reflexive, symmetry, and transitive, will work for Y because the elements of Y are also in X and those properties are satisfied for them when considered as elements of X.

   If Y_i is an equivalence class, say $Y_i = (y)$, then Y_i consists of all $u \in Y$ such that uRy. But these elements u are also in the equivalence class $X_i = (y)$ where we look at the *equivalence class of y in X*. So all such u belong to $Y \cap X_i$. So we have proved that $Y_i \subseteq Y \cap X_i$. Now if any element x belongs to Y and X_i then it also belongs to X_i and hence is related to y because $y \in X_i$ as well. So $x \in Y_i$ as well. So we have proved $Y \cap X_i \subseteq Y_i$ and thus the two sets are equal.

5. For the relation R defined on the set of integers $\mathbb{Z}$ by $mRn \iff 7 \textit{ divides } m - n$ prove the following:

   a) (10 points) Prove that it is an equivalence relation.

   (b) (6 points) Find the equivalence class of 1. Describe the set using set notation.

   (c) (8 points) How many equivalence classes are there? Describe all of them using set notation.

   (d) (7 points) Show that the equivalence classes are distinct and they cover all integers.

   Solution:

   a) Reflexive: $\forall m \in \mathbb{Z}, mRm$ because $7|(m-m)$ because $7|0$.

   Symmetric: $\forall m, n \in \mathbb{Z}, mRn \implies nRm$ because $7|(m-n) \implies 7|(n-m) = -(m-n)$.

   Transitive: $\forall m.n, k \in \mathbb{Z}, mRn, nRk \implies mRk$ because $7|(m-n), 7|(n-k) \implies 7|(m-n+n-k) = m-k$.

   (b) $(1) = \{m \in \mathbb{Z} \mid mR1\} = \{m \in \mathbb{Z} \mid 7|(m-1)\}$. This means all m such that $m - 1 = 7k \implies m = 7k + 1$. So the set is $\{...., -13, -6, 1, 8, 15, ...\}$

   (c) and (d) Similar to (1), you will have $(0), (2), (3), (4), (5), (6)$ so totally 7 equivalence classes. These are all disjoint because basically each equivalence class is the set of integers with a remainder equal to $0, 1, 2, 3, 4, 5, 6$ and an integer can have only one of these as remainders between 0 and 7, Anything outside of these remainders will be in one of the equivalence classes.

6. (Challenge: Extra credit 10 points) Check if the following relation R defined on the set of rational numbers S whose denominators are not divisible by 3 is reflexive, symmetric, and transitive. Here the fractions are written after canceling off common factors. For example,in a/b, a and b have no common factors, and 3 does not divide b and similarly for c/d and all other fractions:

$$\frac{a}{b} R \frac{c}{d} \iff 3 \ divides \ ad - bc.$$

Solution:

Reflexive:

$$\frac{a}{b} R \frac{a}{b} \ because \ ab - ba = 0 \ and \ 3|0.$$

Symmetric:

$$3 \ divides \ ad - bc \implies 3 \ divides \ cb - da = -(ad - bc) \implies \frac{c}{d} R \frac{a}{b}.$$

Transitive:

$$\frac{a}{b} R \frac{c}{d} \ and \ \frac{c}{d} R \frac{e}{f} \implies 3 \ divides \ ad - bc \ and \ 3 \ divides \ cf - de.$$

Multiply $ad - bc$ by f and $cf - de$ by b. Since $3 \ divides \ ad - bc \ and \ 3 \ divides \ cf - de$ we have $3 \ divides \ (ad - bc)f \ and \ 3 \ divides \ (cf - de)b$. This means 3 divides their sum $adf - bcf + bcf - bde = adf - bde = d(af - be)$. This means $3 \ divides \ af - be$ because 3 does not divide d. Therefore we get $\frac{a}{b} R \frac{e}{f}$.

Alternative proof: The condition $3 \ divides \ ad - bc$ is same as saying 3 divides $(a/b) - (c/d)$ because $ad - bc$ is just the numerator of $(a/b) - (c/d)$ and similarly $cd - fe$ is the numerator of $(c/d) - (e/f)$. The key point now is to prove that if 3 divides the numerators of the two fractions $(a/b) - (c/d)$ and $(c/d) - (e/f)$ then it divides the numerator of their sum also. This will be true because 3 does not divide the denominators and is easy to show using algebra. In fact this is true for any two fractions.