5/5/2026 Final Exam Spring 2026,Proofs and Problem Solving Sitaraman

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Instructions: **NO CALCULATORS OR CELLPHONES**

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 120 minutes; Total 100 points.

Please read the questions carefully before answering.

NOTE: IN ALL PROBLEMS BELOW, PROVING JUST A FEW CASES IS NOT ENOUGH. YOU HAVE TO PROVE IN GENERAL, AS IMPLIED BY QUESTIONS.

1. (20 points) Prove by contrapositive: (Graphical solution not acceptable ; Checking a few cases will not get any credit. [Hint : It may help to factor $x^2 - 1$].

$$x^2 - 1 \geq 0 \implies x \leq -1 \text{ } or \text{ } x \geq 1.$$

2. (20 points) Prove if true or give counterexample:

   (a) If x is irrational and y is irrational then xy is irrational.

   (b) If x is rational and y is rational then $x + y$ is rational.

3. Let R be the relation on the set $A = \{f : \mathbb{R} \to \mathbb{R}\}$, the set of real valued functions on the real numbers, defined by $fRg \iff f(x) = kg(x)$ for some fixed real number $k \neq 0$. In other words, one function is a multiple of the other. For example, $\sin x$ and $\pi \sin x$.

   (a) (8 points) Show that this is an equivalence relation.

   (b) (6 points) Find the equivalence class of the function $f(x) = 1$.

   (c) (6 points) Show that $f(x) = x$ and $g(x) = x^2$ belong to different equivalence classes.

4. (20 points) Prove by induction for all natural numbers n and a fixed real number x:

   If $1 + x > 0$ then $(1 + x)^n \geq 1 + nx$.

5. (20 points) One of the following three sets doesn't have the same cardinality as (has no bijection with) the other two. Find which one and prove that there is no bijection for that set with the other two. Also show that the other two are of same cardinality by finding a bijection (1-1, onto map) between them:

   $A = \mathbb{N}$ the set of natural numbers, $B = 2\mathbb{N}$ the set of even numbers, $C =$ set of integer solutions of $x^2 < 64$.

6. (extra credit 20 points) Prove if true or give counterexample: You must use the basic definition of limits. Assume all sequences below consist of only positive real numbers.

   (a) If $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ then $a_n + b_n \to 0$ as $n \to \infty$.

   (b) If $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ then $a_n b_n \to 0$ as $n \to \infty$.