Howard University Math Department

Instructions: **NO CALCULATORS OR CELLPHONES**

PLEASE PROVIDE STEP BY STEP EXPLANATIONS

WRITING ONLY ANSWERS WILL NOT GET FULL CREDIT

Time Limit 120 minutes; Total 100 points.

Please read the questions carefully before answering.

NOTE: IN ALL PROBLEMS BELOW, PROVING JUST A FEW CASES IS NOT ENOUGH. YOU HAVE TO PROVE IN GENERAL, AS IMPLIED BY QUESTIONS.

1. (20 points) Prove by contrapositive:

   Graphical solution not acceptable ; Checking a few cases will not get any credit.

   [Hint : It may help to factor $x^2 - 1$].

   $$x^2 - 1 \geq 0 \implies x \leq -1 \; or \; x \geq 1.$$

   Solution: Need to show:

   $$x > -1 \; AND \; x < 1 \implies x^2 - 1 < 0.$$

   Method 1: $x > -1 \implies (x+1) > 0$ and $x < 1 \implies x - 1 < 0$. But then $(x+1)(x-1) = x^2 - 1 < 0$ because it is a product of positive and negative.

   Method 2: $x > -1 \; AND \; x < 1 \implies |x| < 1$. This means $|x|^2 < 1$. This last statement needs proof. Basically we are saying that if you square a positive number that is between 0 and 1 then you get something that is also less than 1. You can prove it using derivatives (see below). But $|x|^2 = x^2$. So $x^2 < 1$ also.

   Method 3: $x^2 - 1$ is a decreasing function in $(-\infty, 0)$, and an increasing function in $(0, \infty)$ because the derivative $(x^2 - 1)' = 2x$ is positive when x is positive and negative when x is negative.

   Then by substitution we see that $x^2 - 1 = 0$ at -1 and 1. Therefore, for any value between -1 and 1 it is smaller than 0 (i.e, negative). It decreases to -1 at 0 and then increases back to 0 at $x = 1$. So basically at all values between -1 and 1 it is negative, as required to show.

2. (20 points) Prove if true or give counterexample:

   (a) If x is irrational and y is irrational then xy is irrational.

   (b) If x is rational and y is rational then $x + y$ is rational.

   Solution:

(a) This is false. $\sqrt{2} \times \sqrt{2} = 2$ is a counterexample. Here product of two irrational numbers becomes rational.

(b) This is true.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \in \mathbb{Q}.$$

3. Let R be the relation on the set $A = \{f : \mathbb{R} \to \mathbb{R}\}$, the set of real valued functions on the real numbers, defined by $fRg \iff f(x) = kg(x)$ for some fixed real number $k \neq 0$. In other words, one function is a multiple of the other. For example, $\sin x$ and $\pi \sin x$.

   (a) (8 points) Show that this is an equivalence relation.

   (b) (6 points) Find the equivalence class of the function $f(x) = 1$.

   (c) (6 points) Show that $f(x) = x$ and $g(x) = x^2$ belong to different equivalence classes.

   Solution:

   a) Reflexive: $f(x) = 1 \times f(x)$, so fRf for any $f \in A$.

   Symmetric: $f = kg \iff g = (1/k)f$ from the definition of the relation.

   Transitive: $f = kg, g = lh \implies f = (kl)h$. So $fRg, gRh \implies fRh$.

   b) Equivalence class of f is all functions of the form $g(x) = k$ (constant functions) because $gRf \implies g(x) = kf(x) = k$ for some $k \in \mathbb{R}$.

   c) If fRg then $x = kx^2$ for all real x, for some fixed real number k. This means $1 = kx$ for all nonzero x, by canceling one x. This means $x = 1/k$ for all nonzero x. This is a contradiction because k is fixed.

4. (20 points) Prove by induction for all natural numbers n and a fixed real number x:

   If $1 + x > 0$ then $(1 + x)^n \geq 1 + nx$.

   Solution:

   For $n = 1$ we have $1 + x = 1 + x$ so it works.

   Assume true for n, and prove for $n + 1$. In other words, prove $(1 + x)^{n+1} \geq 1 + (n + 1)x$.

   Given $(1 + x)^n \geq 1 + nx$ multiply both sides by $1 + x$.

   Get $(1 + x)^n(1 + x) \geq (1 + nx)(1 + x)$. Note that we are multiplying both sides by a positive number, so the inequality doesn't change direction.

   Now if we prove that $(1+nx)(1+x) \geq 1+(n+1)x$ then we will be done. But $(1+nx)(1+x) = 1 + nx + x + nx^2$ which can be rewritten as $= 1 + (n + 1)x + nx^2$ and this is $\geq 1 + (n + 1)x$ because nx^2 is always nonnegative.

   [This is just the Bernoulli's inequality proved in class].

5. (20 points) One of the following three sets doesn't have the same cardinality as (has no bijection with) the other two. Find which one and prove that there is no bijection for that set with the other two. Also show that the other two are of same cardinality by finding a bijection (1-1, onto map) between them:

$A = \mathbb{N}$ the set of natural numbers, $B = 2\mathbb{N}$ the set of even numbers, $C =$ set of integer solutions of $x^2 < 64$.

Solution: We saw that the map $f(x) = 2x$ gives a bijection from natural numbers to even numbers.

The set $C =$ integer solutions of $x^2 < 64$ is the finite set $\{-8, -7, -6, ..., 6, 7, 8\}$ or basically all the integers from -8 to 8. You cannot have a bijection of a finite set with an infinite set.

6. (extra credit 20 points) Prove if true or give counterexample:

   You must use the basic definition of limits.

   Assume all sequences below consist of only positive real numbers.

   (a) If $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ then $a_n + b_n \to 0$ as $n \to \infty$.

   (b) If $a_n \to 0$ and $b_n \to 0$ as $n \to \infty$ then $a_n b_n \to 0$ as $n \to \infty$.

   Solution: Both are true.

   (a)For any $\epsilon/2 > 0$, $\exists N, M$ such that $|a_n - 0| = a_n < \epsilon/2$ if $n \geq N$ and $|b_n - 0| = b_n < \epsilon/2$ if $n \geq M$. Taking the maximum of M, N we get $|a_n + b_n| = a_n + b_n < (\epsilon/2) + (\epsilon/2) = \epsilon$ if $n > max(M, N)$. So $a_n + b_n \to 0$ also.

   (b) For any $\epsilon > 0$, $\exists N, M$ such that $|a_n - 0| = a_n < 1$ if $n \geq N$ and $|b_n - 0| = b_n < \epsilon$ if $n \geq M$. Taking the maximum of M, N we get $|a_n b_n| = a_n b_n < 1 \times \epsilon = \epsilon$ if $n > max(M, N)$. So $a_n b_n \to 0$ also.