Howard University Math Department

Finding orbit of a relation

We have an invertible (hence 1-1) function from the set $\mathbb{Z}$ to itself given by $f(x) = x + 1$ and a relation on $\mathbb{Z}$ given by

$$xRy \iff f^n(x) = y, \text{ for some } n \in \mathbb{Z}.$$

In other words, y is obtained from x by applying f to x many times, specifically n times. n can be negative also, by applying f^{-1} many times.

Solution:
First we find f^{-1}.

$$f(x) = y = x + 1 \implies x = y - 1 \implies f^{-1}(x) = x - 1.$$

Next we find $f^n(x)$.

$$f^1(x) = x + 1, f^2(x) = f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2,$$

You can see the pattern here. Each time you apply f you increase the value by 1. So $f^n(x) = x + n$.

For example $f^3(2) = f(f(f(2))) = 2 + 3 = 5$.
Similarly $f^{-1}(x) = x - 1, f^{-2}(x) = x - 2, ..., f^{-n}(x) = x - n, ...$
Now the orbit of any $x \in \mathbb{Z}$ would contain all of these, including $f^0(x) = x$.
So the orbit of x is

$$(x) = \{...., x - n, ..., x - 2, x - 1, x, x + 1, x + 2, ..., x + n, ...\} = \mathbb{Z}, \quad \forall x \in \mathbb{Z}.$$

Proving that R is an equivalence relation

Will prove this is an equivalence relation for ANY invertible function f.
Reflexive: $f^0(x) = x$ where f^0 is the identity function, so xRx.
Symmetric: If xRy then $y = f^k(x)$ for some $k \in \mathbb{Z}$. Then $x = f^{-k}(y)$ and $-k \in \mathbb{Z}$ as well.
Transitive: If xRy and yRz then $y = f^k(x)$ for some $k \in \mathbb{Z}$ and $z = f^l(y)$ for some $l \in \mathbb{Z}$ and combining them we get $z = f^l(y) = f^l(f^k(x)) = f^{k+l}(z)$ and $k + l \in \mathbb{Z}$ so xRz as well.