Howard University Math Department

Finding equivalence classes of real numbers modulo 1

We have a relation on $\mathbb{R}$ given by

$$xRy \iff x - y = n, \text{ for some } n \in \mathbb{Z}.$$

In other words, y and x differ by an integer.

Solution:

Basically if you keep adding or subtracting 1 from a real number, you will get all other real numbers that are related to it. For example, 1.1 is related to $0.1, 2.1.3.1, 4.1, ...$ etc on positive side and $-0.9, -1.9, -2.9, ...$ on the negative side.

We say that the numbers are congruent modulo 1.

Proving that R is an equivalence relation

Reflexive: $\forall x \in \mathbb{R}, xRx$ because $x - x = 0$ is an integer.

Symmetric: If xRy then $y = x + k$ for some $k \in \mathbb{Z}$. Then $x = f - k$ and $-k \in \mathbb{Z}$ as well.

Transitive: If xRy and yRz then $y = x + k$ for some $k \in \mathbb{Z}$ and $z = y + l$ for some $l \in \mathbb{Z}$ and combining them we get $z = x + k + l$ and $k + l \in \mathbb{Z}$ so xRz as well.

Equivalence classes

Basically, you can say that the part (the fractional part) after the decimal point in the decimal expansion (for example, 0.1 for 1.1) is all that matters. But these are all the numbers between 0 and 1. So each number x between zero and 1 is the fractional part of a real number of the form $x + n$ where n is an integer. They all give distinct equivalence classes because they are all not related to each other. If $0 \leq x < y < 1$ then $|x - y| < 1$. So $x - y$ cannot be an integer. You can also see that they cover all real numbers but this is automatic once we show that they are all the equivalence classes, because we already proved that for any equivalence relation, the equivalence classes cover the whole set.

ALTERNATE PROOF: This whole problem is covered by the orbits problem we did in previous class because the relation $xRy \iff f^n(x) = y, \text{ for some } n \in \mathbb{Z}$ with $f(x) = x + 1$ is exactly the same as the relation above. Therefore the orbits or equivalence classes are also the same.

$$f^1(x) = x + 1, f^2(x) = f(f(x)) = f(x + 1) = (x + 1) + 1 = x + 2,$$

You can see the pattern here. Each time you apply f you increase the value by 1. So $f^n(x) = x + n$.

For example $f^3(2) = f(f(f(2))) = 2 + 3 = 5$.

Similarly $f^{-1}(x) = x - 1, f^{-2}(x) = x - 2, ..., f^{-n}(x) = x - n, ...$

Now the orbit of any $x \in \mathbb{Z}$ would contain all of these, including $f^0(x) = x$.

So the orbit of x is

$$(x) = \{...., x - n, ..., x - 2, x - 1, x, x + 1, x + 2, ..., x + n, ...\} = \mathbb{Z}, \quad \forall x \in \mathbb{Z}.$$