DeMorgan's Law for Sets

1.
$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

Solution:

For two sets to be equal, need to show each is subset of other.

First show
$$\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}.$$

Need $x \in \overline{A \cup B} \implies x \in \overline{A} \cap \overline{B}$.

Need $x \notin A \cup B \implies x \in \overline{A} \cap \overline{B}$.

But $x \notin A \cup B$ means opposite of $x \in A$ *or* $x \in B$.

This is, by DeMorgan's law for logic, $x \notin A$ *and* $x \notin B$.

This is exactly same as saying $x \in \overline{A}$ and $x \in \overline{B}$.

This is same as $x \in \overline{A} \cap \overline{B}$.

So we have proved $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Next to show
$$\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}.$$

Need to show: $x \notin A$ *and* $x \notin B \implies x \in \overline{A \cup B}$.

Prove by contrapositive: $\neg(x \in \overline{A \cup B}) \implies \neg(x \notin A \textit{ and } x \notin B)$.

This is same as saying: $x \in A \cup B \implies x \in A$ *or* $x \in B$, using DeMorgan's law of logic.

So we have showed $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

2.
$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

Solution: For two sets to be equal, need to show each is subset of other.

First show
$$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}.$$

Need $x \in \overline{A \cap B} \implies x \in \overline{A} \cup \overline{B}$.

Need $x \notin A \cap B \implies x \in \overline{A} \cup \overline{B}$.

But $x \notin A \cap B$ means opposite of $x \in A$ *and* $x \in B$.

This is, by DeMorgan's law for logic, $x \notin A$ *or* $x \notin B$.

This is exactly same as saying $x \in \overline{A}$ or $x \in \overline{B}$.

This is same as $x \in \overline{A} \cup \overline{B}$.

So we have proved $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

Next to show

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}.$$

Need to show: $x \notin A$ *or* $x \notin B \implies x \in \overline{A \cap B}$.

Prove by contrapositive: $\neg(x \in \overline{A \cap B}) \implies \neg(x \notin A$ *or* $x \notin B)$.

This is same as saying: $x \in A \cap B \implies x \in A$ *and* $x \in B$, using DeMorgan's law of logic.

So we have showed $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.