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Easy way to make bijections (Theorem 8.10 in 3rd edition of Lay's book)

Theorem 1. *If there is a surjection* $\mathbb{N} \to S$ *then* S *is countable*

Proof. Suppose f is the surjection. Then make a map $h : S \to \mathbb{N}$ by $h(s)$ going to the smallest natural number n such that $f(n) = s$. Then h is injective because now each $s \in S$ is mapped to a unique pre-image. Also $h(S) \subseteq \mathbb{N}$ and is therefore countable by Theorem 8.9 (in 3rd edition). Since $h(S)$ is countable, there is a bijection $g : h(S) \to \mathbb{N}$. Now it was proved earlier that the composition of two maps is bijective if each one is bijective. So the map $g \circ h : S \to \mathbb{N}$ is a bijection from S to $\mathbb{N}$ and thus S is countable. □

Theorem 2. *If there is an injection* $S \to \mathbb{N}$ *then* S *is countable*

Proof. Suppose f is the injection. Then f is an injection $S \to f(S)$ also. But $f(S) \subseteq \mathbb{N}$ and is therefore countable by theorem 8.9. So there is a bijection $g : f(S) \to \mathbb{N}$ and thus a bijection $g \circ f$ from S to $\mathbb{N}$. Therefore S is countable. □