4-24-2026 Patterns in (environmental) math

Please go to Update page and Course page to see information about class and to keep up to date. The links are in main page of Canvas and also on http://nature-lover.net/math. You can see old notes from spring 25 etc at this website. It will help you prepare for class.

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Today: counting methods and expected value

Repetitions allowed, Permutations, Combinations

Counting number of ways to arrange (or pick): n things in k places

When repetitions are allowed: n x n x …x n = n^k

When repetitions are not allowed (permutations)

and order is important:

$nP_k = \frac{n!}{(n-k)!}$ = n x (n-1) x (n-2) x …x (n-k+1)

(First k numbers starting with n)

Counting number of ways to select (not arrange):

$$nC_k = \frac{nP_k}{k!} = \frac{n!}{(n-k)!\,k!}$$
$$= \frac{\mathrm{n} \times (\mathrm{n}-1) \times (\mathrm{n}-2) \times \ldots . \times (\mathrm{n}-\mathrm{k}+1)}{\mathrm{k}(\mathrm{k}-1)(\mathrm{k}-2) \ldots .3 \times 2 \times 1}$$

Question 1: From 52 cards with four suits clubs, hearts, diamond and spades, how many ways to **select** four spades? What is the probability that a hand of four cards is all spades?

Answer: There are 13 spades. Number of ways to select 4 is 13C4.

13C4 = 13 x 12 x 11 x 10 / (4 x 3 x 2 x 1) = 13 x 11 x 5 = 715.

Follow up question: Suppose you want to find P(4 spades).

Number of favorable outcomes = 715.

Total number of outcomes

= How many hands of 4 cards are possible out of 52?

= 52C4 = 52! /(48! x 4!) = 52 x 51 x 50 x 49 / (4 x 3 x 2 x 1)

P(4 spades) = 715/52C4 = 715/(26 x 17 x 25 x 49)

Expected value and expected winnings

Suppose you throw 2 coins 3 times. HHH, TTT, HTH, … so 2^3 = 8 ways.

Suppose you get 1$ for HHH, 2$ for one tail (HTH, HHT, THH) and 3$ for two tails (HTT, TTH, THT), and 4$ for TTT

What is **expected winnings?**

Add (dollar amounts x Probability of winning that much)

1x P(1) + 2 x P(2) + 3 x P(3) + 4 x P(4)

= 1 x (1/8) + 2 x (3/8) + 3 x (3/8) + 4 x (1/8)

= (1/8)(1+ 6 + 9 + 4) = 20/8 = 2.50 $

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What is the purpose of this?

Supposing 100 people play this game. The total winnings will be quite close to 100 x 2.50 = 250 $. So if the casino charges 3$ for each person, they would make profit of 50$.

Expected value and climate change

We can calculate, using the laws of probability theory, the chance of getting several records in a row.

It turns out to be 1 + (1/2) + (1/3) + …+(1/n)

This is the expected number of record years out of n years.

Compare this to what is happening with the global temperatures.

Question 2: In lottery there are 1000 tickets sold. First prize is 50,000 $, second prize is 10,000$, there are 10 prizes of 100$ each. What are the expected winnings? How much profit would the lottery organizers make if they charge $ 100 for each ticket? (you can calculate the profit two ways).

Maximum they can pay out = 50000+10000+ (10 x 100) = 61000.

if they charge 100 $ for each ticket, then they get 100,000.

So minimum profit would be 39000.

Expected winnings: 61 $ (= 61000/1000).

Using expected value formula:

50000 x P(50K) + 10000 x P(10k) + 100 x P(100)

= 50000 x (1/1000) + 10000 x (1/1000) + 100 x (10/1000)

= 50 + 10 + 1 = 61.

Each ticket wins 61 on average, so totally 61000 is won.

Notice how you get the same number, 61000.

Question 3: In a throw of two dice, each with numbers 1 to 6, find the expected value of the total.

First note that you cannot get a value below 2 or above 12 when throwing two dice. So we only need to find probabilities of the totals being 2, 3, 4,…, 12.

P(2) = P((1,1)) = 1/36.

P(3) = P({(1,2), (2,1)}) = 2/36.

P(4) = P({(1,3), (2,2), (3,1)}) = 3/36.

Similarly P(5) = 4/36, P(6) = 5/36, P(7) = 6/36.

After 7 it starts decreasing.

P(8) = P({(2,6), (3,5), (4,4), (5, 3), (6, 2)}) = 5/36.

Similarly P(9) = 4/36, P(10) = 3/36, P(11) = 2/36, and P(12) = 1/36.

Expected value is the sum of the totals times their probabilities:

= $\sum_{k=2}^{12}\left(k \times P(k)\right) =$ 2 x P(2) + 3 x P(3) + ….+ 11 x P(11) + 12 x P(12)

$$= \frac{2}{36} + \frac{3 \times 2}{36} + \frac{4 \times 3}{36} + \frac{5 \times 4}{36} + \frac{6 \times 5}{36} + \frac{7 \times 6}{36} + \frac{8 \times 5}{36} + \frac{9 \times 4}{36} + \frac{10 \times 3}{36} + \frac{11 \times 2}{36} + \frac{12 \times 1}{36}$$

$$= \frac{2}{36} + \frac{6}{36} + \frac{12}{36} + \frac{20}{36} + \frac{30}{36} + \frac{42}{36} + \frac{40}{36} + \frac{36}{36} + \frac{30}{36} + \frac{22}{36} + \frac{12}{36} = \frac{252}{36} = 7.$$

Practice problems:

1. Find the expected value of the amount you can get if you get 2 dollars for each head, in a toss of 3 coins. Use both methods described above.

2. Find the expected value of a game where you get 1000 dollars for each of the prime numbers between 1 and 10 (basically 2, 3, 5, and 7) and zero for the composite numbers between 1 and 10, when you pick numbers from 1 to 10.