Howard University Math Department

EACH PROBLEM 20 POINTS

1. True or False ? Prove if true and provide counterexample or disprove if false.
   (a) For any $n \in \mathbb{Z}$, n^2+1 is either odd or 2 times an odd number.
   (b) The negative Pell's equation $x^2 - 20y^2 = -1$ has a solution.
   (c) It is possible to find positive integers N such that $\left|\pi - \frac{N}{10^k}\right| < \frac{1}{10^k}$ for any $k = 0, 1, 2, 3, ...$
   (d)
$$\sqrt{1+\sqrt{1+\sqrt{1+.......}}} = \frac{1+\sqrt{5}}{2} \quad (\textit{golden ratio}!)$$

2. Show that $\sqrt{3} = 1.73205...$ is irrational. Find the fraction x/y with $y \le 20$ that gives the closest approximation to $\sqrt{3}$.

3. Find five different solutions to $x^2 - 6y^2 = 3$ using the solutions of $x^2 - 6y^2 = 1$.

4. Using the pigeonhole principle, show that there are at least two positive integers $m, n \in S = \{1, 2, 3, ..., 100\}$ such that $m \equiv n \pmod 9$ and $m \equiv n \pmod{11}$. Then show that there are exactly two such integers. What are they?

5. For the elliptic curve $y^2 = x^3 + 1$ show that $P = (-1, 0)$ and $Q = (0, 1)$ are two rational (actually integer) points.

   (a) Show that $P + Q = R = (2, -3)$, $2P = 0$, and $3Q = 0$ using the geometric addition method (i.e, finding mirror image of third intersection).

   (b) Using only algebra and the 3 relations above, show that $P = -P, 2Q = -Q, P + R = Q, Q + R = P - Q, 2R = -Q, 3R = P$. Also from $P + Q = R$ get that $-R = P - Q$.

   (c) Show that the group generated by P and Q (i.e, $kP + mQ$ for any $m, n \in \mathbb{Z}$) is a cyclic group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ and generated by $P + Q$.

   (Indeed, it is known that the set of *all rational points* is given by the above group! Harder to prove that, though.)