Differential Equations Final Exam Solutions
May 2, 2023
Howard University Mathematics Department

MUST GIVE STEP BY STEP EXPLANATIONS TO GET CREDIT FOR ANSWERS.
No calculators or electronic devices are permitted.
Do ALL PROBLEMS. EACH WORTH 20 POINTS. LAST PROBLEM BONUS.

1. (16 points) Find the integrating factor and the general solution of the following linear equation. (4 points) Give the largest interval over which the general solution is defined.
$$\frac{dy}{dx} = x - y.$$

   Solution: First we write it as $\frac{dy}{dx} + y = x$.

   There is no singular point (coefficient of dy/dx is never zero). So solution will be valid in the interval $(-\infty, \infty)$. So in particular the answer to (b) is $(-\infty, \infty)$.

   (a) Integrating factor is $e^{\int 1dx} = e^x$. Multiplying by it, we get
$$e^x \frac{dy}{dx} + ye^x = xe^x \implies d(ye^x) = xe^x$$

   Integrate RHS by parts:
$$\implies ye^x = xe^x - e^x + C \implies y = x - 1 + \frac{C}{e^x}.$$

2. For the ODE $y'' + y' + y = 0$ find : (12 points) the general solution ; (8 points) the solution satisfying $y(0) = 0, y'(0) = 1$.

   Solution:

   The auxiliary equation is $m^2 + m + 1 = 0$ and it has complex roots: $m = \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i$ using quadratic formula.

   So the general solution is $y = ae^{-x/2}\cos(\sqrt{3}x/2) + be^{-x/2}\sin(\sqrt{3}x/2)$.

   For initial value $y(0) = 0$ we get $0 = a$.

   For initial value $y'(0) = 1$ we get $-(a/2) + (\sqrt{3}b/2) = y'(0) = 1 \implies b = 2/\sqrt{3}$ and so the required solution is
$$y = (2/\sqrt{3})e^{-x/2}\sin(\sqrt{3}x/2).$$

3. (8 points)Write the differential equation for the amount of salt $A(t)$ when the tank initially holds 400 gallons, pure water is pumped in at 4 gallons/min, and salt is pumped out at the same rate.

   (8 points) Solve the ODE and write the formula for $A(t)$.

   (4 points) What happens as $t \to \infty$?

   Solution: When pure water is pumped in, no salt is coming in.

Change in $A(t)$ is only from salt going out.

The rates are same, so amount of solution in tank remains 400 at all times.

Salt is draining out at the rate of $(A(t)/400)$ times 4 so we get the separable ODE

$$\frac{dA}{dt} = 0 - \frac{A(t)}{100} \implies \frac{dA}{dt} = -\frac{A}{100}.$$

$$\frac{dA}{dt} = -\frac{A}{100} \implies \frac{dA}{A} = -\frac{dt}{100} \implies \ln|A| = -\frac{t}{100} + C \implies A(t) = Be^{-t/100}.$$

Here the constant B is actually $\pm e^C$. You see that as $t \to \infty$, $A(t) \to 0$.

4. (12 points) Solve $xy' = 1 + y$ using separable equations method.

   (8 points) Find the solution satisfying $y(1) = 0$.

$$x\frac{dy}{dx} = 1 + y \implies \frac{dy}{1+y} = \frac{dx}{x} \implies \ln|1+y| = \ln|x| + C \implies 1 + y = \pm xe^C.$$

Letting $\pm e^C = A$ we get $y = Ax - 1$.

$y(1) = 0 \implies A = 1$. So required solution is $y = x - 1$.

5. (10 points) Show that $\sin y \; dx + x \cos y \; dy = 0$ is an exact ODE.

   (10 points) Find the solution. Your solution can be of form $f(x, y) = C$.

Solution: $M = \sin y, N = x \cos y$. It is exact because

$$\frac{\partial M}{\partial y} = \cos y, \frac{\partial N}{\partial x} = \cos y.$$

Let LHS equal to $df(x, y)$. Integrating $M = \dfrac{\partial f}{\partial x} = \sin y$ with respect to x, we get $f(x, y) = x \sin y + g(y)$.

Differentiating partially with respect to y we get

$$\frac{\partial f}{\partial y} = x \cos y + g'(y) = N = x \cos y \implies g'(y) = 0 \implies g(y) = B.$$

So $f(x, y) = x \sin y + B = A$ and this can be written as $f(x, y) = x \sin y = C$ by saying $A - B = C$. You can also say $y = \sin^{-1}(C/x)$ is the final solution.

6. (4 points) Is $y' = 1 - \cot(x - y)$ a linear equation? ($\cot(x - y) = \cos(x - y)/\sin(x - y)$)

   (10 points) Make it separable using a substitution.

   (6 points) Find the solution $y(x)$ by solving the separable equation.

Solution:

Actually a non-linear, non-exact, non-separable equation.

The substitution that works here is $u = x - y$.

$$u = x - y \implies \frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \cot(x - y) \implies 1 - \frac{du}{dx} = 1 - \cot u \implies \frac{du}{dx} = \cot u$$

$$du/\cot u = dx \implies (\cos u\, du)/\sin u = dx \implies \ln|\sin u| = x + C \implies \sin u = \pm e^{x+C}$$

$$\implies u = \sin^{-1}(Ae^x) \implies y = x - \sin^{-1}(Ae^x),\ A = \pm e^C.$$

7. Solve $y'' - 4y = x$ using method of undetermined coefficients using the following steps:

   (6 points) Find the complementary solution y_c of the homogenous equation.

   (10 points) Find the particular solution y_p.

   (4 points) Find the general solution.

   Solution:

   The auxiliary equation is $m^2 - 4 = 0$ and it has roots $2, -2$. So we get the solutions e^{2x}, e^{-2x}.

   $y_c = ae^{2x} + be^{-2x}$.

   Let $y_p = Ax + B$. Then $y_p'' - 4y_p = x \implies -4Ax - 4B = x \implies A = -1/4, B = 0$. So $y_p = -x/4$.

   General solution is
   $$y = y_c + y_p = ae^{2x} + be^{-2x} - \frac{x}{4}.$$

8. A mass weighing 24 lbs, attached to a spring that stretched to 4 inches. It is released 3 inches above equilibrium position with downward velocity of 2 ft/s. [Note: 1 foot equals 12 inches].

   (6 points) Find spring constant and ω.

   (10 points) Write the differential equation and find the general solution.

   (4 points) Use initial conditions to find equation of motion.

   Solution: 4 inches is 1/3 foot. From Hooke's law, $24 = k(1/3) \implies k = 72$.

   $m = 24/32 = 3/4 = 0.75$ slugs. (weight is the force due to gravity and it equals mg where m is mass and g is the acceleration due to gravity which is 32 ft/sec/sec).

   $\omega^2 = k/m = 72/(3/4) = 96 \implies \omega = 4\sqrt{6}$.

   We found $\omega = 4\sqrt{6}$. So solution is $x(t) = c_1 \cos(4\sqrt{6}t) + c_2 \sin(4\sqrt{6}t)$.

   Now we use the initial conditions: it is released 3 inches (1/4 foot) above equilibrium position with downward velocity of 2 ft/s.

   So $x(0) = -1/4, x'(0) = 2$.

   This gives $c_1 = -1/4, c_2 = 1/(2\sqrt{6})$. So the solution is $x(t) = -(1/4)cos(4\sqrt{6}t) + (1/(2\sqrt{6}))\sin(4\sqrt{6}t)$.

9. Using power series find two independent solutions for $y'' - x^2y = 0$.

Solution:

$y'' - x^2y = 0$. This has no singular points, the coefficient of y'' is 1 which is defined everywhere.

To solve $y'' - x^2y = 0$ with $y = \sum_{n=0}^{\infty} a_n x^n$, $y'' = \sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2}$

Now plug y, y'' into equation. Then compare coefficients of like powers.

$$\sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} a_n(n(n-1))x^{n-2} = \sum_{n=0}^{\infty} a_n x^{n+2}$$

Let $n-2 = k, n = k+2$ in LHS and $n+2 = k, n = k-2$ in RHS.

$$\implies a_{k+2}(k+2)(k+1) - a_{k-2} = 0, k = 2, 3, 4, ...$$

$$\implies a_{k+2} = a_{k-2}/((k+2)(k+1))$$

We had the recurrence relation

$$a_{k+2} = a_{k-2}/((k+2)(k+1)), \quad k = 2, 3, 4, ...$$

First we note that there is no constant or x term on RHS – it starts with x^2.

So the coefficient of $x^0 = 1$ on LHS is $2a_2$ and it has to be zero.

The coefficient of x is $6a_3$ from LHS, so $a_3 = 0$ as well.

For the other coefficients, using the recurrence relation, we get

$$a_4 = \frac{a_0}{3 \cdot 4}, \quad a_5 = \frac{a_1}{4 \cdot 5}, \quad a_6 = 0, \quad a_7 = 0,$$

$$a_8 = \frac{a_4}{7 \cdot 8} = \frac{a_0}{3 \cdot 4 \cdot 7 \cdot 8}, a_9 = \frac{a_5}{8 \cdot 9} = \frac{a_1}{4 \cdot 5 \cdot 8 \cdot 9}, \quad a_{10} = 0, \ a_{11} = 0, ...$$

Basically all coefficients $a_4, a_8, a_{12}, ...$ are multiples of a_0 and all coefficients $a_5, a_9, a_{13}....$ are multiples of a_1. All of $a_6, a_{10}, a_{14}...$ are zero because they are multiples of a_2 which is 0 and all of $a_7, a_{11}, a_{15}...$ are zero because they are multiples of a_3 which is 0 also.

Separating the two constants, we get the general solution $y = a_0y_1 + a_1y_2$ where $y_1 = 1 + \frac{x^4}{3 \cdot 4} + \frac{x^8}{3 \cdot 4 \cdot 7 \cdot 8} + ...$ and $y_2 = x + \frac{x^5}{4 \cdot 5} + \frac{x^9}{4 \cdot 5 \cdot 8 \cdot 9} + ...$ are two *linearly independent* functions represented by those power series.

10. Solve $y' + y = e^{2t}, y(0) = 0$ using Laplace transforms as follows:

(10 points) Find $L(y)$ by taking Laplace transform of both sides.

(10 points) Find y by taking inverse Laplace transforms.

Solution:

To solve $y' + y = e^{2t}, y(0) = 0$ we use Laplace transform and the formula $L(f'(t)) = -f(0) + sL(f)$.

We have $L(f(t)) = L(y) = F(s), y(0) = f(0) = 0$.

Start by taking Laplace Transform of both sides: $L(y' + y) = L(e^{2t})$. Using linearity $L(y' + y) = L(y') + L(y)$.

Now using formula for $L(y')$ we get

$$sF(s) - f(0) + F(s) = \frac{1}{s-2} \implies F(s) = \frac{1}{(s-2)(s+1)} = \frac{1}{3}\left(\frac{1}{s-2} - \frac{1}{s+1}\right)$$

$$\implies f(t) = L^{-1}F(s) = \frac{1}{3}(L^{-1}(1/(s-2)) - L^{-1}(1/(s+1))) = \frac{e^{2t}}{3} - \frac{e^{-t}}{3}.$$

11. Given the following system:

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

(10 points) Write down characteristic matrix and find eigenvalues

(10 points) Find eigenvectors and general solution.

Solution:

$$AX = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

The characteristic equation is

$$det \begin{pmatrix} -1-\lambda & 0 \\ 1 & -\lambda \end{pmatrix} = 0.$$

Solving, $(-1-\lambda)(-\lambda) = 0 \implies \lambda = -1 \ or \ \lambda = 0$.

$$AX = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} = X'$$

We got the eigenvalues as $\lambda = 0, -1$.

Now we find the eigenvectors for these values:

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = -1 \begin{pmatrix} u \\ v \end{pmatrix} \implies \begin{pmatrix} -1u \\ 1u \end{pmatrix} = \begin{pmatrix} -1u \\ -1v \end{pmatrix}$$

Comparing both sides in last equation, we see that $-1u = -1u, -1v = 1u$.

The first equation tells us nothing. But the second equation says $-1v = 1u \implies -v = u$.

So u can be anything, $v = -u$ means one eigenvector is $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

All other eigenvectors for -1 will be multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and we call $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ as K.

Note that eigenvector for 0, which we will call L, will be different from that of -1 *!*

Now we find the eigenvectors for 0:

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0 \begin{pmatrix} u \\ v \end{pmatrix} \implies \begin{pmatrix} -1u \\ 1u \end{pmatrix} = 0$$

Comparing both sides in last equation, we see that $-1u = 0, 1u = 0$.

There is no condition on v but the equation says $1u = 0 \implies u = 0$.

So an eigenvector for 0 is $L = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

All other eigenvectors for 0 will be multiples of $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ We got, for the eigenvectors,

$$K = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The solution for the system is $c_1 K e^{-t} + c_2 L e^{0t} = c_1 K e^t + c_2 L$.

12. (10 points) Find the differential equation for the following model: The rate of change of the velocity $v(t)$ of an object of mass m attracted by an object of mass M is proportional to the mass M and inversely proportional to the square the distance $x(t)$ between them. Note that M is fixed.

    (5 points) Convert the ODE into a second order equation involving x and t only.

    (5 points) Is the second order equation linear? Can it be solved using auxiliary equation?

    Solution:

$$\frac{dv}{dt} \propto \frac{M}{x^2} \implies \frac{dv}{dt} = \frac{kM}{x^2}$$

where k is a constant. Let $v = x'$ we get $x^2 x'' = kM$. This is not linear and we cannot use auxiliary equation method.