3-23-2020 Notes, Proofs 2

Differential Equations – Epidemics

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Outline

Review of basic exponential growth and decay model

Whenever there is growth / decay in nature, such as **population growth, radioactive decay, cooling of objects, ...** an exponential function is lurking behind.

$P(t)$ is population at time t, $P(0)$ is initial population, k is called "growth rate."

Equation: $P(t) = P(0)e^{kt}$

$k > 0 \rightarrow$ Growth ; $k < 0 \rightarrow$ Decay.

Properties of exponential function

What is the derivative of $P(t)$ given by

$$P(t) = P(0)e^{kt} \text{ ?}$$

Basic exponential function – rate of growth / decay

$$P'(t) = P(0)(ke^{kt}) = k(P(0)e^{kt}) = kP(t).$$

So the rate of growth at any instant is
proportional to the value at that instant !!

In fact, ONLY functions of the form Ce^{kt} have this property! Proof below.

k is called the **relative growth rate.**

NOTE: This works only for uninhibited growth. Any real life situation, such as the population of an animal species within an island, would be restrained by space, food availability, disease, conflict, etc.,

Differential Equation for Exponential Function

We saw earlier,

$$P'(t) = P(0)(ke^{kt}) = k(P(0)e^{kt}) = kP(t).$$

If $y = P(t)$, this can be written as

$$\frac{dy}{dt} = ky.$$

This is a *separable* differential equation, and it can be solved as

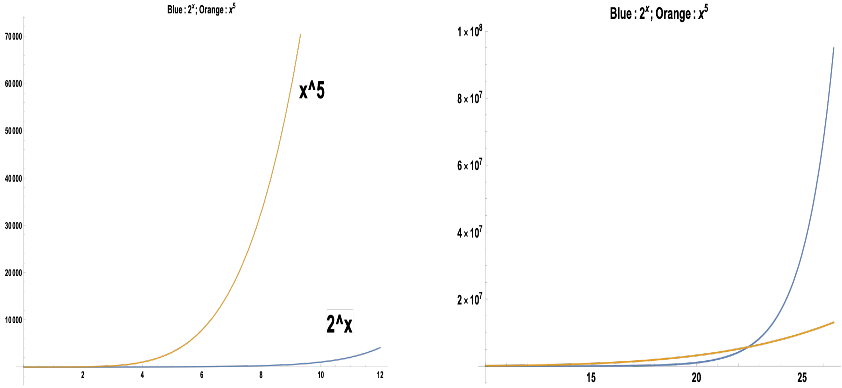
$$\frac{dy}{y} = kdt \implies \int \frac{dy}{y} = \int k\,dt \implies \ln y = kt + C \implies y = e^{kt}e^{C}$$

Letting $e^C = P(0)$, we get $P(t) = P(0)e^{kt}$.

This proves that ONLY functions of the form $P(0)e^{kt}$ have the property $y' = ky$.

Explosive nature of exponential growth

Compared to polynomial functions, exponential functions seem to grow slowly at first, but then explode.



What if population growth is not uninhibited?

LOGISTIC MODEL OF POPULATION GROWTH
Equation for usual, *unrestricted* population growth:

$$P(t) = P(0)e^{kt}, k > 0.$$

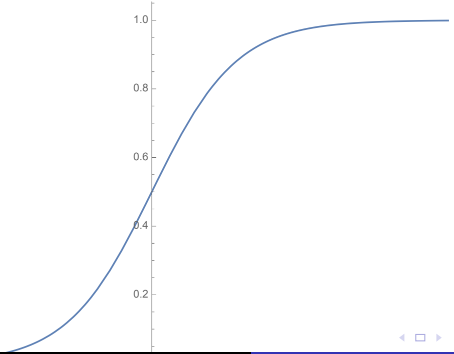
But usually in real life after growing fast initially, populations tend to be restricted by amount of resources available and also disease, etc.,
The *Logistic model* is one such realistic model.

Example graph of logistic population growth

Example plot of a logistic graph :

$$P(t) = \frac{1}{1 + e^{-0.5t}}$$

As time goes population stabilizes at 1.
(Ignore the negative side because time is always positive).



Differential Equation for Logistic model

The function $P(t) = 1/(1 + e^{-0.5t})$ is a solution for the differential equation $\dfrac{dy}{dx} = (0.5)(1 - y)y$.
Notice that when y is small, $1 - y \simeq 1$ and it is almost same as $y' = (0.5)y$. This is of the form $y' = ry$, the equation from which the exponential function $y = e^{rt}$ comes out. So the function has a graph similar to the exponential function when $y = P(t)$ is small, as you can see in its graph.
You can also see that the maximum value is 1 (as $t \to \infty, \; e^{-0.5t} \to 0, \; P(t) \to 1$.)
1 is called the carrying capacity of this function. When $y \to 1, 1 - y \to 0$ so derivative $dy/dx = (0.5)(1 - y)y \to 0$, i.e, slope approaches zero, and graph is almost horizontal, as you can see.

Differential Equation for general Logistic model

The general differential equation for functions with logistic growth is $\frac{dy}{dt} = r(1 - \frac{y}{M})y$ where M is the carrying capacity. In example above, $M = 1, r = 0.5$.

Solution of the differential equation: (Partial Fractions Method)

$$\frac{dy}{dt} = r(1 - \frac{y}{M})y \implies \int \frac{dy}{(1 - \frac{y}{M})y} = \int r\ dt$$

$$\implies \frac{1}{M} \int \frac{dy}{1 - (y/M))} + \int \frac{dy}{y} = rt + C$$

$$\implies -\ln(1 - y/M) + \ln y = rt + C \implies \frac{y}{1 - \frac{y}{M}} = e^{rt} e^{C}$$

Differential Equation for general Logistic model – 2

Solving for y and letting $e^C = A$, a constant, we get

$$y = \frac{Ae^{rt}}{1 + \dfrac{Ae^{rt}}{M}}.$$

Solving for A you can show that $A = \dfrac{y(0)}{1 - (y(0)/M)}$.

Here $y(0)$ is the population at $t = 0$.

In our example we have $y(0) = P(0) = 1/(1 + e^{-0.5(0)}) = \dfrac{1}{2}$.

Plugging this for A and $M = 1$ we get $A = \dfrac{1/2}{1 - (1/2)} = 1$.

So the equation is

$$y = \frac{e^{0.5t}}{1 + e^{0.5t}} = \frac{1}{1 + e^{-0.5t}}.$$

The S.I.R model

(Adapted from MAA website of the same title).
We divide the population into three groups.
We also assume that the total population remains fixed at N.
There are no births or immigrants coming in or emigrants going out.
$S = S(t)$ is the number of **susceptible (vulnerable to being infected)** individuals at time t, $I = I(t)$ is the number of **infected** individuals at time t, and $R = R(t)$ is the number of **recovered** individuals at time t.
("recovered" really means those who can neither catch the infection nor pass on the infection any more. It includes those who got infected and developed immunity and those who got infected and died).

S.I.R model – set up

Also let $s(t) = S(t)/N$, the susceptible fraction of the population, $i(t) = I(t)/N$, the infected fraction , and $r(t) = R(t)/N$, the recovered fraction.
At any given time, people who are susceptible are becoming infected, and people who are infected are recovering.
So **all three quantities** s, i and r are **changing with respect to time.**
We need to find out how they are changing, i.e, their derivative.
In other words, we will make differential equations for all three.
Then we can write down expressions for them by solving the differential equations.

S.I.R model – set up – continued

But unlike the simple functions we saw earlier, we **won't get a single differential equation** that is easy to solve using integration.
This is because all of them influence each other.
Instead we get a **system of three differential equations** that we will see how to solve approximately, using numerical methods.

Derivative of $s(t)$

First we write a differential equation for the number of susceptible people.
We need to know how that number is changing.
We assume the following:

1. The only way an individual leaves the susceptible group is by becoming infected.
2. The time-rate of change of $S(t)$, the number of susceptibles, depends on the number already susceptible, the number of individuals already infected, and the amount of contact between susceptibles and infecteds.

Derivative of $s(t)$ – equation

Suppose each infected individual has a fixed number b of contacts per day that are sufficient to spread the disease. Not all these contacts are with susceptible individuals. If we assume a uniform mixing of the population, the fraction of these contacts that are with susceptibles is $bs(t)$. Thus, on average, each infected individual generates $bs(t)$ new infected individuals per day. [With a large susceptible population and a relatively small infected population, we can ignore tricky counting situations such as a single susceptible encountering more than one infected in a given day.] Since there are a total of $I(t)$ infected people, we get $b \times s(t) \times I(t)$ susceptible people getting infected, and so $S(t)$ *decreases* by that much. This results in

$$\frac{dS}{dt} = -bs(t)I(t).$$

Derivative of $s(t)$ – example

Dividing both sides by N we get

$$\frac{ds}{dt} = -bs(t)i(t). \tag{1}$$

For example, if $S = 100$, $I = 10$, $R = 10$ then $N = 120$ and if each infected person meets 2 people (so $b = 2$) then the fraction of them who are capable of being infected is $2 \times s(t) = 2 \times (100/120) = 5/3$. Since there are 10 infected people, totally they spread the disease to $10 \times (5/3) = 50/3 \simeq 17$ people. So the next day there will be $100 - 17 = 83$ susceptible people. There will also be $10 + 17 = 27$ infected people minus the number of people who moved from infected to "recovered' category. This will give us the next equation.

Derivative of $r(t)$ and $i(t)$

A certain number of infected people become "recovered" each day, and that is the only way R changes.
So we get

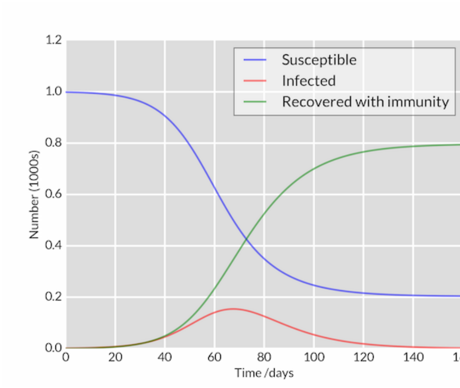
$$\frac{dR}{dt} = kI(t) \implies \frac{dr}{dt} = ki(t). \tag{2}$$

Now from $S + I + R = N$ we get $s + i + r = 1$ and from this we get $\frac{ds}{dt} + \frac{dr}{dt} + \frac{di}{dt} = 0$.
Solving for di/dt, and plugging in ds/dt and dr/dt from the two differential equations (1) and (2),

$$\frac{di}{dt} = bs(t)i(t) - ki(t). \tag{3}$$

Solving the differential equations

We can solve the 3 equations (1), (2), (3) together approximately using differentials and linear approximation.
Use $f(a+h) = f(a) + f'(a)h$ repeatedly.
Plug in initial values, say s close to 1 and i, r close to 0, get values of $s'(0), r'(0), i'(0)$, find $s(h), r(h), i(h)$ and so on.

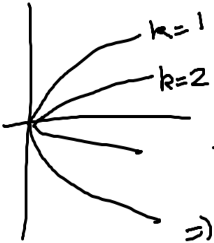


Exercises

1. Use Euler's method (the one using linear differentials) with step size $h = 0.1$ to find approximate values for $y = f(x)$ given that $y' = x + y,\ y(0) = 1$.
2. Try getting solutions using initial values of s, r, i as above and Euler's method.
3. Find the differential equation satisfied by the family of parabolas $x = ky^2$. You must get an equation valid for all such parabolas. In other words, it must not have k in it.
4. A model for measuring learning is given by $\dfrac{dP}{dt} = k(M - P)$ where $P(t)$ is the performance after training for t days, M is the maximum level of performance, $k > 0$ is a constant. Solve this for $P(t)$. What is the limit as $t \to \infty$?

Solution for Problem 3 – page 1

$$x = ky^2 \Rightarrow \frac{x}{y^2} = k$$



$$\Rightarrow \left(\frac{x}{y^2}\right)' = 0$$

$$\Rightarrow \frac{y^2 - x(2yy')}{y^4} = 0$$

$$\Rightarrow y^2 - x\left(2y\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x}$$

$$x^2 + y^2 = r^2$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

No r involved!

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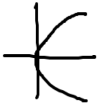
Solution for Problem 3 – page 2

$$\int \frac{2\,dy}{y} = \int \frac{dx}{x}$$

(Original eqn $x = ky^2$)

$$\Rightarrow 2\int \frac{dy}{y} = \int \frac{dx}{x}$$



$$\Rightarrow 2\ln y = \ln x + C$$

$$\Rightarrow \ln y^2 = \ln x + C$$

$$\Rightarrow y^2 = x e^C \Rightarrow y^2 e^{-C} = x$$

$$\Rightarrow x = ky^2 \quad (k = e^{-C})$$