Howard University Math Department

1. How do climate scientists know how much carbon there was over the past thousands of years?

   Answer: See notes from 3/6/19.

2. Find the sum of the first 20 terms of $1 + \frac{1}{3} + \frac{1}{9} + ...$

   Calculate as much as you can but show how you get each step.

   What happens if you try to calculate the infinite sum?

   Soln: The first term $a = 1$. The common ration $R = 1/3$.

   The sum of the first 20 terms is

   $$A = a\left(\frac{1-R^n}{1-R}\right) = 1\left(\frac{1-\left(\frac{1}{3}\right)^{20}}{1-\frac{1}{3}}\right) \simeq \frac{3}{2}(1-(3\times 10^{-10})) \simeq 1.5$$

   You can see that the infinite sum will approach 1.5 because $(1/3)^{20}$ gets smaller and smaller and so after a while the sum is almost equal to 1.5.

3. Travis decided to put \$320 in his savings account at the end of every month. Find the amount he has at the end of 8 years, if the money is worth 7 percent annual rate compounded monthly.

   Solution:

   Use the formula $A(t) = \frac{m((1+i)^{nt}-1)}{i}$ where $A(t)$ is amount after t years, r is the annual rate, and $i = r/n$. m is monthly payment/investment, n is number of times interest is compounded (here $n = 12$)

   [We are really using formula $a(1-R^n)/(1-R)$ for sum of geometric series with $R = 1+i$ because we are adding the total amount coming out of *each* monthly payment, and each of those is $m(1+i)^T$ for whatever time period T it sits in the account].

   Here $m = 320, r = 0.07, n = 12, t = 8$.

   We get $A(t) = 320\frac{(1+(.07/12))^{96}-1}{.07/12} = 41023.62$ dollars.

4. In July 2005, the internet was linked by a global network of about 352.8 million host computers. The number of host computers has been growing approximately exponentially and was about 35.3 million in July 1998.

   (a) Let t be the number of years since July 1998 (so in July 20005 it was 7). Find a formula for the number, *N,* of internet host computers (in millions of computers) as an exponential function of *t,* using exponential function of the form $N(t) = ae^{kt}$. What are the values of a and k in your model?

   [Hint: Use this equation for $N(7)$ with $N(7) = 352.8$ and solve for k].

(b) What is the doubling time of N?

Solution: a) Since $N = 35.3$ when $t = 0$, we have $N = 35.3e^{kt}$. We use the fact that $N = 352.8$ when $t = 7$ to find k: $N = 35.3e^{kt} \implies 352.8 = 35.3e^{k\cdot 7} \implies \frac{352.8}{35.3} = e^{7k} \implies \ln(352.8/35.3) = 7k \implies k = \frac{\ln(352.8/35.3)}{7} \approx 0.3289$

b) We find t when $N = 2(35.3) = 70.6 \implies 70.6 = 35.3e^{0.3289t} \implies 2 = e^{0.3289t} \implies \ln 2 = 0.3289t \implies t = \frac{\ln 2}{0.3289} \approx 2.107.$

The number of internet host computers is doubling every 2.107 years.

Correct Answers:

- `a = 35.3, k = 0.32886`
- `2.10773`

5. Write the repeating decimal 0.77777.... as a fraction using the formula for an infinite gemetric sum.

   Solution: Use $S = a/(1-r)$ formula.

$$0.7777... = \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + ... = \frac{7/10}{1-\frac{1}{10}} = \frac{7/10}{9/10} = \frac{7}{9}.$$

6. What percentile value do you assign to the 85 degree temperature reading in the following list of average daily temperatures from July 2015 in DC?

   73,75,77,78,78,79,79,79,79,80,80,80,81,81,82,82,83,83,83,83,83,83,83,85,85,85,85,86,86,89,90

   Solution:

   85 appears as the 24th, 25th, 26th and 27th values. So we take it to be the 23+ (4/2) = 25th value. So its percentile rank is 25/31 = 0.8064 = 81 percent approximately which gives 81st percentile.

7. Write a frequency table, calculate the mean (average), median, mode and standard deviation of above data.

   Solution:

   The frequency table will have three rows, one for the 70's, one for the 80's and one for 90's.

   Note that there are 31 readings.

   Since the values are already in ascending order, you just need to find the middle (16th) value to get the median.

   Clearly 83 is the mode since it appears 7 times, more than any other reading.

   To find standard deviation, first find average, then find difference of each value from average, then add the squares of all the averages, divide this sum by 31 and then find the square root of the result.

8. One incoming Howard University freshman was ranked 10th in a class of 125; another ranked 75th in a class of 620. Which has the higher percentile rank?

   Solution:

   10th rank of 125 means 115 students are below this score. So percentile rank is $115/125 =$ 92nd percentile.

   75th of 620 means 545 are below, corresponding to $545/620 =$ 88th percentile.

   So 10th of 125 is the higher percentile rank.

   You see how percentile ranking helps to compare ranks from different size populations.

9. (Problem about recycling) Given X tonnes of virgin pulp on any day, 3/4 of that pulp is made into paper. Suppose also that 3/4 of paper is recycled in the city. After the first day 3/4 of the paper gets recycled, and since the paper used is 3/4 of the X amount of pulp, the amount of pulp (coming from the recycled paper) is $3/4 \times 3/4 \times X = (9/16) \times X$ and so on. For the next day we again have X tonnes of new pulp plus $(9/16) \times X$ tonnes of pulp from recyled paper giving $X + (9/16) \times X$. For the third day we will have $X + (9/16)(pulp\ from\ second\ day) = X + (9/16)(X + (9/16)X) = X + (9/16)X + (9/16)^2X$.

   Go through the process and write an expression for *total* amount of pulp coming from recycled paper after n days. If this process goes on infinitely, how much pulp comes from X amount of the original pulp?

   Solution:

   The total amount of pulp for the second day will be X amount of new pulp plus $9/16 \times X$ amount of pulp from recycled paper for a total of $X + (9/16)X$.

   For the third day it will be $X+(9/16)(pulp\ from\ second\ day) = X+(9/16)(X+(9/16)X) = X+(9/16)X+(9/16)^2X$. and so on. The "infinite" sum will look like $X+(9/16)X+(9/16)^2X+(9/16)^3X + ... = X/(1-(9/16)) = X \times (16/7)$. Here we used the formula $a/(1-r)$ for the infinite sum with $a = X$ and $r = 9/16$.

   [In turn, $\dfrac{a}{1-r}$ comes from $a\left(\dfrac{1-r^n}{1-r}\right)$ and $r^n \to 0$ as $n \to \infty$].

10. In July 2005, the internet was linked by a global network of about 352.8 million host computers. The number of host computers has been growing approximately exponentially and was about 35.3 million in July 1998.

    (a) Find a formula for the number, *N,* of internet host computers (in millions of computers) as an exponential function of *t,* the number of years since July 1998, using exponential function of the form $N(t) = ae^{kt}$. What are the values of a and k in your model?

    (b) What is the doubling time of N?

    Solution: a) Since $N = 35.3$ when $t = 0$, we have $N = 35.3e^{kt}$. We use the fact that $N = 352.8$ when $t = 7$ to find k: $N = 35.3e^{kt} \implies 352.8 = 35.3e^{k\cdot 7} \implies \frac{352.8}{35.3} = e^{7k} \implies \ln(352.8/35.3) = 7k \implies k = \frac{\ln(352.8/35.3)}{7} \approx 0.3289$

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11. All organic matter (plants, animals,?) contain two types of carbon. Carbon-12 which is regular carbon, and carbon 14 which is radioactive and decays over time. The amount of carbon 14 left in an organic object can be calculated using the ratio of carbon-12 to carbon-14. The equation for the amount left $A(t)$ is given by the following equation :

$$A(t) = A(0)e^{-t/8033}$$

Here $A(0)$ is the original amount, $1/8033 = 0.0001245$is the decay constant. Notice the negative sign in the exponent. If it were exponential growth, the exponent would be positive.

(a) How much carbon-14 would be left after 1000 years if you start with 100g ?

(b) Find the half-life of carbon-14 (how long would it take for, say, 100g to become 50g? It doesn?t matter how much you start with, though).

Solution:

(a) $A(1000) = 10e^{-1000/8033} = 8.83g.$

(b) Half-life is given by

$$A(t) = \frac{1}{2}A(0) \implies A(0)e^{-t/8033} = \frac{1}{2}A(0) \implies e^{-t/8033} = 1/2$$

$$\implies -t/8033 = \ln(0.5) \implies t = -8033 \times \ln(0.5) = 5568.05\ years.$$

You can also use the formula for half-life (or doubling time) $T = \dfrac{\ln 2}{k}$ where k is rate of decay (or growth). Here $k = 1/8033$. Also note that $\ln(1/2) = \ln(2^{-1}) = -\ln 2$.