Howard University Math Department

1. Find the 1000th term of the sequence $1, -2, 4, -8, \ldots$. You can leave your answer as a power.

   Solution:

   The common ratio is -2 in this geometric sequence.

   The n-th term is $a_n = a_1 r^{n-1} = 1(-2)^{n-1} = (-2)^{n-1}$.

   Therefore $a_{1000} = (-2)^{999} = -(2^{999})$ because 999 is odd.

   You can leave the answer like that.

2. Consider the quadratic function $w(x) = -3x^2 - 15x + 27$.

   (a) The vertex of this parabola is at what point?

   (Enter a point as (h, k) including parentheses. Give exact coordinates, not approximations.)

   (b) What is the axis of symmetry for this parabola?

   (Enter your answer in the form of an equation for a vertical line of symmetry, not just a number.)

   Solution:

   Axis of symmetry is at
   $$x = -b/2a = -\frac{-15}{2(-3)} = -\frac{5}{2}.$$

   Plugging $-5/2$ into the expression for y we get
   $$y = \frac{183}{4}.$$

   Thus, the vertex of w is
   $$\left(-\frac{5}{2}, \frac{183}{4}\right)$$
   and the axis of symmetry is the vertical line $x = -\frac{5}{2}$ through the vertex.

3. The height y (in feet) of a ball thrown by a child is $y = -\frac{1}{12}x^2 + 6x + 5$ where x is the horizontal distance in feet from the point at which the ball is thrown.

   (a) How high is the ball when it leaves the child's hand? (Hint: Find y when $x = 0$.)

   (b) What is the maximum height of the ball?

   (c) How far from the child does the ball strike the ground? (Hint: Find where $y = 0$).

   Correct Answers:

- `5`
- `113`
- `(6*12+sqrt(6^2*12^2+4*5*12))/2 = 72.83`

Solution:

(a) Put $x = 0$ in $\frac{1}{12}x^2 + 6x + 5$ to get $y = 5$.

(b) To get the maximum height we need to find the vertex.

First note that $a = -1/12, b = 6, c = 5$. Because a is negative, the parabola will be facing down.

The x value of the vertex is given by $-b/2a = -6/(2\times(-1/12)) = -6/(-1/6) = -6\times -6 = 36$.

Put $x = 36$ in $y = \frac{1}{12}x^2 + 6x + 5$ to get the y value of the vertex.

You get $y = (-1/12)(36^2) + 6(36) + 5 = -108 + 216 + 5 = 113$.

This is the maximum height because the parabola is facing down (down because a is negative).

(c) To find the point where it hits ground, put $y = 0$ and solve for x.

$$\frac{-1}{12}x^2 + 6x + 5 = 0 \implies x^2 - 72x - 60 = 0 \text{ (mulitply by -12)}$$

$$\implies x = \frac{-(-72) \pm \sqrt{(-72)^2 - 4(1)(-60)}}{2} \text{ (quadratic formula) } = 72.83 \text{ or } -.83$$

The correct answer is the positive number because distance cannot be negative.

4. For the following sequences, find the formula for the $n-th$ term and the value of the fifth term using the formulae for the arithmetic sequence or geometric sequence depending on what it is.

(a) $6, \frac{24}{5}, \frac{96}{25}, \frac{384}{125}, \ldots$

(b) $-1, 3, 7, 11, \ldots$

Correct Answers:

- `6*(4/5)^(n-1)`
- `1536/625`
- `-1+(n-1)4 = 4n-5`
- `15`

(a) This is a geometric sequence because each time we are multiplying by the same number 4/5.

So the common ratio is $r = 4/5$ and the first term $a = 6$.

You can get the fifth term simply by multiplying the fourth by 4/5. You get 1536/125.

Using the formula for $n-th$ term we get $a_n = ar^{n-1} = 6(4/5)^{n-1}$.

(b) This is an arithmetic sequence because each time we are adding the same number 4.

So the common difference is $d = 4$ and the first term $a = -1$.

You can get the fifth term simply by adding 4 to the fourth. You get 15.

Using the formula for $n - th$ term we get $a_n = a + d(n-1) = -1 + 4(n-1) = 4n - 5$.

5. In July 2005, the internet was linked by a global network of about 352.8 million host computers. The number of host computers has been growing approximately exponentially and was about 35.3 million in July 1998.

(a) Find a formula for the number, *N*, of internet host computers (in millions of computers) as an exponential function of *t*, the number of years since July 1998, using exponential function of the form $N(t) = ae^{kt}$. What are the values of a and k in your model?

(b) What is the doubling time of N?

Solution: a) Since $N = 35.3$ when $t = 0$, we have $N = 35.3e^{kt}$. We use the fact that $N = 352.8$ when $t = 7$ to find k: $N = 35.3e^{kt} \implies 352.8 = 35.3e^{k \cdot 7} \implies \frac{352.8}{35.3} = e^{7k} \implies \ln(352.8/35.3) = 7k \implies k = \frac{\ln(352.8/35.3)}{7} \approx 0.3289$

b) We find t when $N = 2(35.3) = 70.6 \implies 70.6 = 35.3e^{0.3289t} \implies 2 = e^{0.3289t} \implies \ln 2 = 0.3289t \implies t = \frac{\ln 2}{0.3289} \approx 2.107$.

The number of internet host computers is doubling every 2.107 years.

Correct Answers:

- `a = 35.3, k = 0.32886`
- `2.10773`

6. The following formulas give the populations (in 100's) of four different cities where t is in years. For each equation, decide whether the relationship is exponential, linear, or neither. In the case of exponential function say what is the growth rate, i.e, by what percentage is the population growing or declining each year. In the case of linear function say by what amount the population is increasing or decreasing.

(a) $P_A = 29(1.25)^t$

(b) $P_B = 1000 + 0.86t$

(c) $P_C = 1750(0.98)^t$

(d) $P_D = 70e^{0.1t}$

Solution:

(a) The formula $P_A = 29(1.25)^t$ is exponential and it is growing by 25 percent per year.

(b) The formula $P_B = 1000 + 0.86t$ is linear and it is growing at a constant rate of 86 people (0.86 times 100) per year.

The equation is of the form $y = mx + b$.

(c) The formula $P_C = 1750(0.98)^t = 1750(1 - 0.02)^t$ is exponential and it is shrinking by 2 percent per year.

(d) The formula $P_D = 70e^{0.1t}$ is exponential and it is growing by 10 percent per year because $r = 0.1$.

7. The rat population in a major metropolitan city is given by the formula $n(t) = 62e^{0.025t}$ where t is measured in years since 1991 and n is measured in millions.

   (a) What was the rat population in 1991?

   (b) What is the rat population going to be in the year 2005?

   Solution:

   Plug in $t = 0$ and $t = 14$ because 1991 is the starting time and $2005 = 1991 + 14$.

   (a) Put $t = 0$ in the formula. You get $n(0) = 62e^{0.025\times 0} = 62e^0 = 62$ millions or 6.2×10^7.

   (b) Put $t = 14$ in the formula. You get $n(14) = 62e^{0.025(14)} = 62e^{0.35} = 87.982$ millions or 8.7982×10^7.

8. In 2010, the population of a country was 80 million and growing at a rate of 1.4 per year. Assuming the percentage growth rate remains constant, express the population, P, in millions,as a function of t, the number of years after 2010.

   Solution:

   The population is growing at a rate of 1.4 per year. So, at the end of each year, the population is $100 + 0.014 = 101.4$ of what it had been the previous year. The growth factor is 1.014. If P is the population of this country, in millions, and t is the number of years since 2010, then, after one year,

$$\begin{aligned} & P = 80(1.014). \\ \text{After two years,} \quad & P = 80(1.014)(1.014) = 80(1.014)^2 \\ \text{After three years,} \quad & P = 80(1.014)(1.014)(1.014) = 80(1.014)^3 \\ \text{After } t \text{ years,} \quad & P = 80\underbrace{(1.014)(1.014)\ldots(1.014)}_{t \text{ times}} = 80(1.014)^t \end{aligned}$$

9. Match each story with a formula. Assume that the constants P_0, r, b, A are all positive.

   1. The amount of charge on a capacitor in an electric circuit decreases by 30 percent each second.
   2. In 1950, the population of a town was 3000 people. Over the course of the next 50 years, the town grew at a rate of 10 percent per decade.
   3. The percent of a lake's surface covered by algae, initially at 35 percent, was halved each year since the passage of anti-pollution laws.
   4. Polluted water is passed through a series of filters. Each filter removes all but 30 percent of the remaining impurities from the water.
   5. In 1950, the population of a town was 3000 people. Over the course of the next 50 years, the town grew at a rate of 250 people per year.

A. $f(x) = B(0.3)^x$

B. $f(x) = A(2)^{-x}$

C. $f(x) = P_0(1 + r)^x$

D. $f(x) = B(0.7)^x$

E. $f(x) = P_0 + rx$

Correct Answers:

1 D

2 C

3 B

4 A

5 E

Explanation:

In general, if something increases or decreases by a percentage or multiple of *the amount at THAT time* then it is modeled by an exponential function.

If it increases or decreases by the same amount each time it is a linear function.

1. Charge decreasing by 30 percent each time means it is an exponential function with equation that looks like $A(t) = P(1 + r)^t$ with $r = -0.3$. Notice that for decreasing functions the rate is negative. So $A(t) = B(1 - 0.3)^t = B(0.7)^t$ is the right answer.

[Whether we call it t or x it doesn't matter].

2. Increases by 10 percent each time means its equation looks like $P(t) = P(1+r)^t = P(1.1^t)$.

3. Halved each time means it is modeled by $A(t) = P(1/2)^x = P(2^{-x})$.

4. Each time all but 30 percent is removed means pollution is decreased by 70 percent each time. So $r = -.7$ and equation looks like $A(t) = P(1 - .7)^t = P(.3)^t$.

5. This needs a linear equation because each time we are just adding 250 people.